THE LINEAR PARAMETERS AND THE DECOUPLING MATRIX FOR LINEARLY COUPLED MOTION IN 6 DIMENSIONAL PHASE SPACE

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Abstract

It will be shown that starting from a coordinate system where the 6 phase space coordinates are linearly coupled, one can go to a new coordinate system, where the motion is uncoupled, by means of a linear transformation. The original coupled coordinates and the new uncoupled coordinates are related by a 6×6 matrix, R. It will be shown that of the 36 elements of the 6×6 decoupling matrix R, only 12 elements are independent. A set of equations is given from which the 12 elements of R can be computed from the one period transfer matrix. This set of equations also allows the linear parameters, the β_i, α_i, $i = 1, 3$, for the uncoupled coordinates, to be computed from the one period transfer matrix.

1 THE DECOUPLING MATRIX, R

The particle coordinates are assumed to be x, p_x, y, p_y, z, p_z. The particle is acted upon by periodic fields that couple the 6 coordinates. The linearized equations of motion are assumed to be

$$\begin{aligned} \frac{dx}{ds} &= A(s)x \\ x &= \begin{bmatrix} x \\ p_x \\ y \\ p_y \\ z \\ p_z \end{bmatrix}, \end{aligned} \tag{1-1}$$

where the 6×6 matrix $A(s)$ is assumed to be periodic in s with the period L. Note that the symbol x is used to indicate both the column vector x and the first element of this column vector. The meaning of x should be clear from the context. The 6×6 transfer matrix $\mathrm{T}(s, s_0)$ obeys

$$\begin{aligned} x(s) &= \mathrm{T}(s, s_0)x(s_0) \\ \frac{d\mathrm{T}}{ds} &= A(s)\mathrm{T} \end{aligned} \tag{1-2}$$

It is assumed that the motion is symplectic so that

$$\mathrm{T}\overline{\mathrm{T}} = I, \quad \overline{\mathrm{T}} = \tilde{\mathrm{S}}\tilde{\mathrm{T}}\,\mathrm{S} \tag{1-3}$$

where I is the 6×6 identity matrix, $\tilde{\mathrm{T}}$ is the transpose of T and the 6×6 matrix S is given by

$$\mathrm{S} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{bmatrix} \tag{1-4}$$

The 6×6 transfer matrix $\mathrm{T}(s, s_0)$ has 36 elements. However, the number of independent elements is smaller because of the symplectic conditions given by Eq. (2-3). There are 15 symplectic conditions or $(k^2 - k)/2$ where $k = 6$. The transfer matrix T then has 21 independent elements.

One can also introduce the one period transfer matrix $\hat{\mathrm{T}}(s)$ defined by

$$\hat{\mathrm{T}}(s) = \mathrm{T}(s + L, s) \tag{1-5}$$

$\hat{\mathrm{T}}(s)$ is also symplectic and has 21 independent elements.

One now goes to a new coordinate system where the particle motion is not coupled. The coordinates in the uncoupled coordinate system will be labeled u, p_u, v, p_v, w, p_w. It is assumed that the original coupled coordinate system and the new uncoupled coordinate system are related by a 6×6 matrix $R(s)$

$$\begin{aligned} x &= R\,u \\ u &= \begin{bmatrix} u \\ p_u \\ v \\ p_v \\ w \\ p_w \end{bmatrix} \end{aligned} \tag{1-6}$$

$R(s)$ will be called the decoupling matrix.

One can introduce a 6×6 transfer matrix for the uncoupled coordinates called $P(s, s_0)$ such that

$$u(s) = P(s, s_0)u \tag{1-7}$$

and one finds that

$$P(s, s_0) = R^{-1}(s)\mathrm{T}(s, s_0)R(s_0) \tag{1-8}$$

one can also introduce the one period transfer matrix $\hat{P}(s)$ defined by

$$\begin{aligned} \hat{P}(s) &= P(s + L, s) \\ \hat{P}(s) &= R^{-1}(s + L)\hat{\mathrm{T}}(s)R(s) \end{aligned} \tag{1-9}$$

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The decoupling matrix is defined as the 6×6 matrix that diagonalize $\hat{P}(s)$, which means here that when the 6×6 matrix $\hat{P}$ is written in terms of 2×2 matrices it has the form

$$\hat{P} = \begin{bmatrix} \hat{P}_{11} & 0 & 0 \\ 0 & \hat{P}_{22} & 0 \\ 0 & 0 & \hat{P}_{33} \end{bmatrix} \tag{1-10}$$

where $\hat{P}_{ij}$ are 2×2 matrices. $\hat{P}$ will be called a diagonal matrix in the sense of Eq. (1-10).

The definition given so far of the decoupling matrix R, will be seen to not uniquely define R and one can add the two conditions on R that it is a symplectic matrix and it is a periodic matrix. The justification for the above is given by the solution found for $R(s)$ below.

Because $\mathrm{T}(s, s_0)$ and $R(s)$ are symplectic, it follows that $P(s, s_0)$ and $\hat{P}(s)$ are symplectic. Eq. (1-8) can be rewritten as

$$\begin{aligned} P(s, s_0) &= \overline{R}(s)\ \mathrm{T}(s, s_0)\ R(s_0) \\ \hat{P}(s) &= \overline{R}(s)\ \hat{\mathrm{T}}(s)\ R(s) \end{aligned} \tag{1-11}$$

It also follows that the 2×2 matrices has 3 independent elements as $|\hat{P}_{11}| = |\hat{P}_{22}| = |\hat{P}_{33}| = 1$. Eq. (1-12) can be written as

$$\hat{\mathrm{T}}(s) = R(s)\ \hat{P}(s)\ \overline{R}(s) \tag{1-12}$$

Eq. (1-12) shows that R has 12 independent elements, as $\hat{\mathrm{T}}$ has 21 independent elements and $\hat{P}$ has 9 independent elements. As R has only 12 independent elements, one can suggest that R has the form, when written in terms of 2×2 matrices,

$$R = \begin{bmatrix} q_1 I & R_{12} & R_{13} \\ R_{21} & q_2 I & R_{23} \\ R_{31} & R_{32} & q_3 I \end{bmatrix} \tag{1-13}$$

where q_1, q_2, q_3 are scalar quantities, the R_{ij} are 2×2 matrices and I is the 2×2 identity matrix. The matrix in Eq. (1-13) appears to have 27 independent elements. However, R is symplectic and has to obey the 15 symplectic conditions, and this reduces the number of independent elements to 12. The justification for assuming this form of R, given by Eq. (1-13), will be provided below where a solution for R will be found assuming this form for R.

Using Eq. (1-13) for R and the symplectic conditions, one can, in principle, solve Eq. (1-12) for R and $\hat{P}$ in terms of the one period matrix $\hat{\mathrm{T}}$. This was done by Edwards and Teng[1] for motion in 4-dimensional phase space where $\hat{\mathrm{T}}$ has 10 independent elements, R has 4 independent elements and $\hat{P}$ has 6 independent elements. An analytical solution of Eq. (1-12) for the 6-dimensional case was not found. However, a different procedure for finding $\hat{P}$ and R will be given by finding the eigenvectors of $\hat{P}$, using the eigenvectors of the one period matrix, $\hat{\mathrm{T}}$.

The 2×2 matrices P_{11}. P_{22}, P_{33} which make up $\hat{P}$ each have 3 independent elements and can be written in the form

$$\begin{aligned} \hat{P}_{11} &= \begin{bmatrix} \cos\psi_1 + \alpha_1 \sin\psi_1 & \beta_1 \sin\psi_1 \\ -1/\gamma_1 \sin\psi_1 & \cos\psi_1 - \alpha_1 \sin\psi_1 \end{bmatrix} \\ \gamma_1 &= (1 + \alpha_1^2)/\beta_1 \end{aligned} \tag{1-14}$$

with similar expressions for $\hat{P}_{22}$ and $\hat{P}_{33}$ Eq. (1-14) and the similar expressions for $\hat{P}_{22}$, $\hat{P}_{33}$ can be used to define the three beta functions β_1, β_2 and β_3.

2 THE LINEAR PARAMETERS β, α, AND ψ AND THE EIGENVECTORS OF THE TRANSFER MATRIX

In this section, the eigenvectors of the one period transfer matrix, $\hat{P}$, will be found and expressed in terms of the linear periodic parameters β, α and ψ. These will be used below to compute the linear parameters from the one period transfer matrix $\hat{\mathrm{T}}$. They will also be used to find the three emittance invariants ϵ_1, ϵ_2 and ϵ_3 and to express them in terms of the linear parameters β_i, α_i, $i = 1, 3$.

The uncoupled transfer matrix obeys

$$\begin{aligned} \frac{d}{ds} &= P(s, s_0) = B(s)\ P(s, s_0) \\ B &= \overline{R}\ A\ R + \frac{d\overline{R}}{ds} \end{aligned} \tag{2-1}$$

This follows from Eq. (1-2) and Eq. (1-11).

One sees from Eq. (2-1) that $B(s)$ is a periodic matrix, $B(s+L) = B(s)$. It can also be shown that B is a periodic, diagonal matrix similar to $\hat{P}$. See [6] for details.

As the 2×2 matrix B_{11} is periodic, one can show[2] that the eigenvector of the transfer matrix for $\overline{u}$ is

$$\begin{aligned} \overline{u}_1 &= \begin{bmatrix} \beta_1^{1/2} \\ \beta_1^{1/2}(-\alpha_1 + i) \end{bmatrix} \exp(i\psi_1) \\ \tilde{\tilde{u}}_1^*\ S\ u_1 &= 2i \end{aligned} \tag{2-2}$$

with the eigenvalue $\lambda_1 = \exp(i\mu_1)$. $\beta_1(s)$, $\alpha_1(s)$ are periodic functions and the phase function $\psi_1 = \mu_1 s/L + g_1(s)$ where $g_1(s)$ is periodic.

One can now write down the eigenvectors of the $\hat{P}$ matrix using Eq. (2-2). These eigenvectors will be called $u_1, u_2, u_3, u_4, u_5, u_6$, each of which is a 6×1 column vector with the eigenvalues $\lambda_1 = \exp(i\mu_1)$, $\lambda_3 = \exp(i\mu_2)$, $\lambda_5 = \exp(i\mu_3)$, $\lambda_2 = \lambda_1^*$, $\lambda_4 = \lambda_3^*$ and $\lambda_6 = \lambda_5^*$.

3 COMPUTING THE LINEAR PARAMETERS β, α, ψ FROM THE TRANSFER MATRIX

An important problem in tracking studies is how to compute the linear parameters, β, α, ψ, defined in section 3, from the one period transfer matrix. The one period transfer matrix can be found by multiplying the transfer matrices of each of the elements in a period. A procedure is given below for computing the linear parameters, which also computes the decoupling matrix R from the one period transfer matrix.

The first step in this procedure is to compute the eigenvectors and their corresponding eigenvalues for the one period transfer matrix $\hat{\mathrm{T}}$. This can be done using one of the standard routines available for finding the eigenvectors of a real matrix. $\hat{\mathrm{T}}$ is assumed to be known. In this case, there are 6 eigenvectors indicated by the 6 column vectors x_1, x_2, x_3, x_4, x_5 and x_6. Because $\hat{\mathrm{T}}$ is a real 6×6 matrix, $x_2 = x_1^*$, $x_4 = x_3^*$, $x_6 = x_5^*$. The corresponding eigenvalue for x_1 is $\lambda_1 = \exp(i\mu_1)$ and the eigenvalue for x_2 is $\lambda_1^* = \exp(i\mu_1)$. In a similar way, λ_2, λ_2^* are the eigenvalues for x_3 and x_4, and λ_3, λ_3^* are the eigenvalues for x_5 and x_6. One can show that (see [6] for details).

$$\begin{aligned} \psi_1 &= ph(x_1) \\ 1/\beta_1 &= Im(p_{x1}/x_1) \\ \alpha_1 &= -\beta_1 Re(p_{x1}/x_1) \end{aligned} \tag{3-1}$$

where Im and Re stand for the imaginary and real part, and ph indicates the phase.

Using Eq. (3-1), one can find the linear parameters β_1, α_1, and ψ_1 from the eigenvector x_1 of $\hat{\mathrm{T}}$. A procedure can be given for computing the entire R matrix. See [6] for details.

4 THE THREE EMITTANCE INVARIANTS

Three emittance invariants will be found for linear coupled motion in 6-dimensional phase space. Expressions will be found for these invariants in terms of β_i, α_i. A simple and direct way to find the emittance invariants is to use the definition of emittance suggested by A. Piwinski[4] for 4-dimensional motion. This is given by

$$\epsilon_1 = |\ \widetilde{x}_1\ \ S\, x|^2 \tag{4-1}$$

x is a 6×1 column vector representing the coordinates x, p_x, y, p_y, z, p_z. x_1 is a 6×1 column vector which is an eigenvector of the one period transfer matrix $\hat{\mathrm{T}}$. x_1 is assumed to be normalized so that

$$\widetilde{x}_1^*\ \ S\, x_1 = 2i \tag{4-2}$$

One first notes that ϵ_1 given by Eq. (4-1) is an invariant since $\widetilde{x}_1\ \ S\, x$ is a Lagrange invariant as x_1 and x are both solutions of the equations of motion. Eq. (3-1) then represents an invariant which is a quadratic form in x, p_x, y, p_y, z, p_z. This result can be expressed in terms of the linear parameters β_1, α_1 by evaluating ϵ_1 in the coordinate system of the uncoupled coordinates. Since the uncoupled coordinates, represented by the column vector u, is related to x by the symplectic matrix R,

$$\epsilon_1 = |\widetilde{u}_1\ S\, u|^2 \tag{4-3}$$

u_1 is an eigenvector of the one period matrix $\hat{P}$, and one sees that because of Eq. (1-11),

$$x_1 = R\, u_1 \tag{4-4}$$

one can now use the result for u, given by Eq. (2-5) and find that

$$\begin{aligned} \epsilon_1 &= \frac{1}{\beta_1}\left[(\beta_1 p_u + \alpha_1 u)^2 + u^2\right] \\ \epsilon_1 &= \gamma_1 u^2 + 2\alpha_1 u p_u + \beta_1 p_u^2 \\ \gamma_1 &= (1+\alpha_1)^2/\beta_1 \end{aligned} \tag{4-5}$$

5 REFERENCES

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