SYNCHROTRON OSCILLATION DRIVEN BY RF PHASE NOISE

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Abstract

We report results of calculations and measurements relating RF phase noise to longitudinal motion of a stored beam. Treating the beam as a noise driven coupled oscillator system, we have made calculations to determine what coupled bunch synchrotron oscillation amplitudes result from RF phase noise. Measurements have also been carried out at CESR of phase noise in the RF system and coupled bunch synchrotron oscillation amplitudes. We also consider the impact of this noise on the dynamic range of a longitudinal feedback system.

1 UNCOUPLED SINGLE BUNCH RESPONSE

The equation of motion for small phase oscillations ϕ in an accelerator is given by [1]

$$\ddot{\phi} + 2\alpha_s\dot{\phi} + \Omega_s^2\phi = 0, \tag{1}$$

where α_s is the damping decrement given by

$$\alpha_s = -\frac{1}{2T_0}\left.\frac{dU}{dE}\right|_{E_0} \tag{2}$$

and Ω_s is the synchrotron frequency given by

$$\Omega_s^2 = \frac{-h\omega_0\eta e}{cp_0T_0}\left.\frac{dV}{d\psi}\right|_{\psi_s}, \tag{3}$$

where T_0 is the revolution time, ω_0 is the revolution frequency with $h\omega_0 = \omega_{RF}$, and η is the momentum compaction. For a sinusoidal driving potential,

$$V(\psi) = V_0\sin\psi = V_0\sin(\psi_s + \phi), \tag{4}$$

where V_0 is the amplitude of the driving potential and ψ_s is the synchronous phase, Eq. 3 becomes

$$\Omega_s^2 = \frac{-h\omega_0\eta}{cp_0T_0}eV_0\cos\psi_s. \tag{5}$$

1.1 Noise response

An RF noise error, δV, can add either a noise forcing term (additive noise) or frequency fluctuations (multiplicative noise) or both in Eq. 1. For amplitude noise,

$$\delta V = v(t)\sin(\psi), \tag{6}$$

where $v(t)$ is a stochastic amplitude error. For phase noise,

$$\delta V = V_0(\sin(\psi + \theta(t)) - \sin(\psi)), \tag{7}$$

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where $\theta(t)$ is a stochastic phase error. For small $\theta(t)$ this can be approximated by

$$\delta V \approx V_0\,\theta(t)\cos\psi. \tag{8}$$

The equation of motion including both additive and multiplicative noise is

$$\ddot{\phi} + 2\alpha_s\dot{\phi} + (\Omega_s^2 + g(t))\phi = f(t), \tag{9}$$

where $f(t)$ is the additive noise term and $g(t)$ is the multiplicative noise term. For amplitude noise

$$f_A(t) = \frac{h\omega_0\eta}{cp_0T_0}e\,v(t)\sin\psi_s \tag{10}$$

$$g_A(t) = \frac{h\omega_0\eta}{cp_0T_0}e\,v(t)\cos\psi_s, \tag{11}$$

and for phase noise

$$f_\phi(t) = \frac{h\omega_0\eta}{cp_0T_0}eV_0\,\theta(t)\cos\psi_s \tag{12}$$

$$g_\phi(t) = \frac{h\omega_0\eta}{cp_0T_0}eV_0\,\theta(t)\sin\psi_s. \tag{13}$$

Either RF amplitude noise or phase noise can produce both additive and multiplicative noise in the equation of motion for ϕ. Though in general amplitude noise will contribute more to multiplicative noise, and phase noise will contribute more to additive noise.

1.1.1 Additive noise

The complex frequency response, $H(\omega)$, to a noise driving force (additive noise only) can be found by making the substitution $\phi(t) = H(\omega)\exp(-i\omega t)$ and $f(t) = \exp(-i\omega t)$ in the equation of motion to obtain [2]

$$H(\omega) = \frac{-1}{\omega^2 - \Omega_s^2 + 2i\alpha_s\omega}. \tag{14}$$

The mean square response is then given by

$$\langle\phi^2\rangle = \int_{-\infty}^{\infty}|H(\omega)|^2 S_f(\omega)d\omega = \int_{-\infty}^{\infty}\frac{S_f(\omega)}{(\Omega_s^2 - \omega^2)^2 + 4\alpha_s^2\omega^2}d\omega \tag{15}$$

where $S_f(\omega)$ is the mean square spectral density of the excitation (noise). The spectral density is the Fourier transform of the autocorrelation function of the excitation, $R_f(\tau)$, i.e.

$$S_f(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty}R(\tau)e^{-i\omega\tau}d\tau \tag{16}$$

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where

$$R(t-t') = \langle f(t)f(t')\rangle \quad (17)$$

where the brackets denote an average over the ensemble of realizations of f. For white noise, $S_f(\omega) = S_f =$ constant,

$$\langle\phi^2\rangle = \frac{\pi S_f}{2\alpha_s\Omega_s^2}. \quad (18)$$

1.1.2 Multiplicative noise

The derivation of the response for a combination of additive and multiplicative noise is a bit more involved. The result given is only good for zero-centered, delta-correlated white noise where the spectral density of $f(g)$ is $S_{f(g)}(\omega) = S_{f(g)} =$ constant. From references [3, 4] the mean square response is given by

$$\langle\phi^2\rangle = \frac{\pi S_f}{2\alpha_s\Omega_s^2 - \pi S_g}. \quad (19)$$

Comparing this to Eq. 18 one can see that the frequency fluctuations act to increase the the mean square phase oscillations. This can be interpreted as an effective increase of the spectral density of the additive fluctuations, $S_f \to S_f/(1-\pi S_g/2\alpha_s\Omega_s^2)$. Also note that Eq. 19 only makes sense for $S_g < \pi\alpha_s\Omega_s^2$. For $S_g > \pi\alpha_s\Omega_s^2$, large frequency fluctuations produce an energy instability that grows exponentially in time.

2 COUPLED BUNCH RESPONSE

For a single bunch of charge Ne in a circular accelerator, the equation of motion for small phase oscillations including coupling due to wake fields is [5]

$$\ddot\phi + 2\alpha_s\dot\phi + \Omega_s^2\phi = \frac{Nr_0c\eta}{\gamma T_0}\sum_{k=0}^{\infty} W_0''(-kC)(\phi(t)-\phi(t-kT_0)). \quad (20)$$

For additive noise forcing we can solve this in the same manner as in Sec. 1.1.1 for the spectral response. In terms of the longitudinal impedance, the spectral response is given by

$$H(\omega) = -\left[\omega^2 - \Omega_s^2 + 2i\alpha_s\omega + \left(\frac{iNr_0\eta}{\gamma T_0^2}\right)\Xi\right]^{-1} \quad (21)$$

where

$$\Xi = \sum_{p=-\infty}^{\infty}[p\omega_0 Z_0^{\parallel}(p\omega_0) - (p\omega_0+\omega)Z_0^{\parallel}(p\omega_0+\omega)]. \quad (22)$$

3 RF CAVITY COUPLING

The cavity itself is also coupled to a generator that drives the voltage in the cavity. It was convenient to measure the noise of the generator rather than the noise in the RF cavities. Therefore it was important to know how the noise in the generator is transferred to the beam via the RF cavities. We can treat this as a system comprised of two coupled oscillators, the cavity and the beam. Consider two oscillators cascaded where one is driven and there is no feedback from the second oscillator to the first, having equations of motion

$$\ddot x + 2\alpha_x\dot x + \Omega_x^2 x = f(t) \quad (23)$$

$$\ddot y + 2\alpha_y\dot y + \Omega_y^2 y = x(t) \quad (24)$$

It is easy to show that in this case the spectral response of the second oscillator is

$$H_y(\omega) = -\frac{H_x(\omega)}{\omega^2-\Omega_y^2+2i\alpha_y\omega} = H_{x_0}(\omega)H_{y_0}(\omega) \quad (25)$$

where H_{x_0} and H_{y_0} denote the spectral response functions for the uncoupled oscillators. The mean square response is then just the combination of the uncoupled response functions with the spectral density of the excitation

$$\langle y^2\rangle = \int_{-\infty}^{\infty}|H_{x_0}(\omega)H_{y_0}(\omega)|^2 S_f(\omega)\,d\omega. \quad (26)$$

A similar case arises for the system we are considering when there is little beam loading on the RF cavities; the motion of the beam is driven purely by the voltage in the RF cavity. The spectral response of the RF cavity is given by the impedance of the cavity and the spectral response of the beam is given in Eq. 14. In this case, the total response of the beam is given by

$$H(\omega) = -\frac{Nr_0\eta}{\gamma T_0^2}\frac{(\omega_{RF}+\omega)Z_0^{\parallel}(\omega_{RF}+\omega)}{\omega^2-\Omega_s^2+2i\alpha_s\omega}. \quad (27)$$

For an RF cavity impedance

$$Z_0^{\parallel}(\omega) = \frac{R_s}{1+iQ\left(\frac{\omega_r}{\omega}-\frac{\omega}{\omega_r}\right)}, \quad (28)$$

where ω_r is the cavity resonant frequency, Eq. 15 gives the mean square response as

$$\langle\phi^2\rangle \approx \left(\frac{Nr_0\eta}{\gamma T_0^2}\right)^2\frac{\pi S_f}{4\alpha_s\Omega_s^2}Z^2, \quad (29)$$

where $Z^2 = |Z_0^{\parallel}(\omega_{RF}+\Omega_s)|^2 + |Z_0^{\parallel}(\omega_{RF}-\Omega_s)|^2$ is approximately given by

$$Z^2 \approx \omega_{RF}^2 R_s^2\left(\frac{1}{1+4\frac{Q^2}{\omega_r^2}(\Omega_s-\Delta\omega)^2} + \frac{1}{1+4\frac{Q^2}{\omega_r^2}(\Omega_s+\Delta\omega)^2}\right), \quad (30)$$

where $\Delta\omega = \omega_{RF}-\omega_r$.

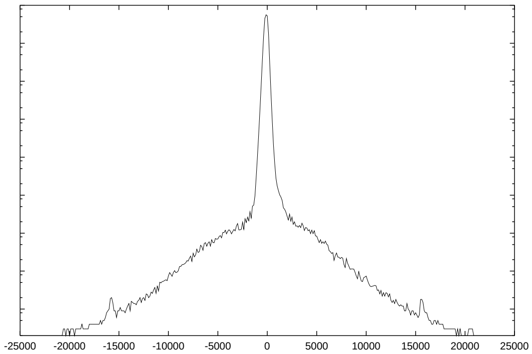

Figure 1: The RF generator spectrum about $\omega_{RF} = 499.765$ MHz.

4 RESULTS

4.1 Measurements

At CESR, measurements were made of the noise spectrum of the RF generator and of the synchrotron oscillation amplitude. The measured synchrotron oscillation amplitude was 2.7 mrad. The noise spectral density of the RF generator at the synchrotron frequency was measured to be 1.7×10^{-11} rad^{2}Hz^{-1}. From Eq. 29, this would correspond to a synchrotron oscillation amplitude of 1.9 mrad. Improvements were then made to reduce the noise in the RF generator. After the improvements were made the measured noise spectral density was 6.3×10^{-12} rad^{2}Hz^{-1}. This would correspond to a predicted synchrotron oscillation amplitude of 1.1 mrad compared to a measured amplitude of 1.2 mrad.

CESR Parameters	
E_0	5.3 GeV
T_0	2.56 μs
γ	10370
η	0.01
α	1160 s^{-1}
Ω_s	20 kHz
ω_{RF}	499.765 MHz
ω_r	499.750 MHz
R_s	140 MΩ
Q	6275
N	10^{11}

Table 1: The CESR parameters used for the calculations of synchrotron oscillation amplitudes.

4.2 Longitudinal feedback

A receiver for a longitudinal feedback system might compare the phase of the beam to a reference oscillator. The resulting error signal would then be sent to a power amplifier which would apply a voltage to the beam. The damping rate of a such a feedback system is

$$\alpha_E < \frac{\delta E \eta}{2\pi E_0 Q_s \sigma_\tau} \quad (31)$$

where σ_τ is the rms phase fluctuation divided by the RF frequency, and δE is the average voltage applied to the beam, equal to half the maximum voltage for a linear feedback system.

The phase fluctuation in this case can come from either the beam motion or the reference oscillator. We have found that the reference oscillator phase fluctuations are much larger than the beam phase fluctuations, so σ_τ=2.5 ps. We are constructing a feedback system for CESR with a maximum voltage of 1.5 kV, so the maximum damping rate α_E is 1730 s^{-1}.

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6 REFERENCES

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