NONLINEAR DYNAMICS OF SINGLE BUNCH INSTABILITY

G. V. Stupakov
Stanford Linear Accelerator Center, Stanford University, P.O. Box 4349, Stanford, CA 94309 USA,
B.N. Breizman and M.S. Pekker
Institute for Fusion Studies, The University of Texas, Austin TX, 78712, USA

Abstract

A nonlinear equation is derived that governs the evolution of the amplitude of unstable oscillations with account of quantum diffusion effects due to the synchrotron radiation. Numerical solutions to this equation predict a variety of possible scenarios of nonlinear evolution of the instability some of which are in good qualitative agreement with experimental observations.

1 INTRODUCTION

Microwave single bunch instability in circular accelerators has been observed in many machines. The instability usually arises when the number of particles in the bunch exceeds some critical value, N_c, which varies depending on the parameters of the accelerating regime.

Recent observations on the SLC damping rings at SLAC [1] with a new low-impedance vacuum chamber revealed new interesting features of the instability. In some cases, after initial exponential growth, the instability eventually saturated at a level that remained constant through the accumulation cycle. In other regimes, relaxation-type oscillations were measured in nonlinear phase of the instability. In many cases, the instability was characterized by a frequency close to the second harmonic of the synchrotron oscillations.

Several attempts have been made to address the nonlinear stage of the instability [2, 3, 4] based on either computer simulations or some specific assumptions regarding the structure of the unstable mode. An attempt of a more general consideration of the problem is carried out in this paper. We adopt an approach recently developed in plasma physics for analysis of nonlinear behavior of weakly unstable modes in dynamic systems [5]. Assuming that the growth rate of the instability is much smaller than its frequency, we find a time dependent solution to Vlasov equation and derive an equation for the complex amplitude of the oscillations valid in the nonlinear regime. Numerical solutions to this equation predict a variety of possible scenarios of nonlinear evolution of the instability some of which are in good qualitative agreement with experimental observations.

2 BASIC EQUATIONS

We start from the equations of motion in longitudinal direction (see, e.g., Ref. [6]):

$$\dot{z} = -c\eta\delta, \quad \dot{\delta} = K(z,t), \tag{1}$$

where z is the longitudinal coordinate, δ is the relative energy deviation, η is the slip factor, the dot indicates differentiation with respect to time t, and

$$K(z,t) = \frac{\omega_{s0}^2}{\eta c} z - \frac{r_e}{T_0\gamma}\int_z^\infty dz' n(z',t)\, w(z'-z). \tag{2}$$

In Eq. (2), ω_{s0} denotes the unperturbed synchrotron frequency, T_0 is the revolution period, r_e is the classical electron radius, γ is the relativistic factor, $n(z,t)$ is the longitudinal beam density, $\int_{-\infty}^{\infty} n(z,t)\,dz = N$, where N is the number of particles in the bunch, and $w(z)$ is the longitudinal wake function. The first term in Eq. (2) corresponds to the potential of the accelerating voltage, and the second term describes the wakefield generated by the bunch.

The distribution function $\psi(z,\delta,t)$ satisfies the Vlasov equation with a Fokker-Planck "collision" term on the right hand side,

$$\frac{\partial\psi}{\partial t} + \{H,\psi\} = R, \tag{3}$$

where we have the Poisson brackets on the left hand side, H is the Hamiltonian corresponding to the equations of motion Eq. (1), and R describes the effect of the synchrotron radiation,

$$R = \frac{\partial}{\partial\delta}\left(\gamma_D\psi\delta + \kappa\frac{\partial\psi}{\partial\delta}\right). \tag{4}$$

In Eq. (4), γ_D is the damping time for the amplitude of the synchrotron oscillations, and κ is the diffusion coefficient associated with the quantum nature of the radiation. In the equilibrium state, the distribution function ψ is given by Haissinski solution,

$$\psi(z,\delta) = \text{const}\times\exp\left(-H_0(z,-\delta)/c\eta\sigma_E^2\right), \tag{5}$$

where $\sigma_E = \sqrt{\kappa/\gamma_D}$ is the rms energy spread of the beam in the absence of the wake, and H_0 is the equilibrium Hamiltonian.

It is convenient to introduce dimensionless variables, $x = z/\sigma_z$, $p = -\delta/\sigma_E$, $\tau = t\omega_{s0}$, and $F = \sigma_z\psi$, where σ_z is the rms length of the beam without wake, $\sigma_z = \sigma_E|\eta|c/\omega_{s0}$. In these variables, the Hamiltonian H takes the form

$$H(x,p,\tau) = \frac{1}{2}p^2 + U(x,\tau), \tag{6}$$

where the "potential energy" U is

$$U = \frac{1}{2}x^2 - I\int_x^\infty dx' S(x'-x)\int_{-\infty}^\infty dp F(x',p,\tau), \tag{7}$$

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with

$$I = \frac{Nr_e}{T_0 \gamma \omega_{s0} \sigma_z \sigma_E}, \quad (8)$$

and $S(x) = \int_0^{x\sigma_s} dz w(z)$. Note that the function S is a dimensionless function of its argument.

Let us perform a canonical transform from x and p to action and angle variables J and θ, of the equilibrium Hamiltonian H_0, and denote by $\tilde{V}$ the deviation of the potential energy from the equilibrium, $\tilde{V} = U - U_0$. Since H_0 depends on J only, the total Hamiltonian $H(\theta, J, t)$ takes the form

$$H(\theta, J, \tau) = H_0(J) + \tilde{V}(\theta, J, \tau). \quad (9)$$

The Vlasov equation for F in terms of action-angle variables is

$$\frac{\partial F}{\partial \tau} + \omega_s \frac{\partial F}{\partial \theta} + \frac{\partial \tilde{V}}{\partial J}\frac{\partial F}{\partial \theta} - \frac{\partial \tilde{V}}{\partial \theta}\frac{\partial F}{\partial J} = R, \quad (10)$$

where $\omega_s = \omega_s(J)$ is the frequency of synchrotron oscillations with the wake taken into account, $\omega_s(J) = dH_0/dJ$.

3 LINEAR THEORY

Suppose that $F_0(J)$ is the equilibrium distribution function, and $\delta F(J, \theta, \tau) = F - F_0(J)$ is its deviation from the equilibrium. In linear theory, $\delta F = f_1(J, \theta) e^{-i\omega\tau} + \text{c.c.}$, where the notation " c.c." denotes a complex conjugate to the first term. The perturbation of the potential $\tilde{V}$ is $\tilde{V} = V_\omega e^{-i\omega\tau} + \text{c.c.}$. Since V_ω is a periodic function of θ, we can expand it in Fourier series, $V_\omega = \sum_{n=-\infty}^{\infty} v_n(J) e^{in\theta}$. For simplicity, we will neglect here the effect of the synchrotron damping in the linear theory by dropping the R-term in Eq. (10). This greatly simplifies the linear analysis and is usually assumed in the literature. However, it can be shown that the effect of the synchrotron damping is crucial for the nonlinear stage of the instability and will later be included in the derivation of the nonlinear equations.

Substituting the expressions for δF and $\tilde{V}$ into Eq. (10) gives in linear approximation

$$-i\omega f_1 + \omega_s \frac{\partial f_1}{\partial \theta} = F_0' \sum_{n=-\infty}^{\infty} inv_n(J) e^{in\theta}, \quad (11)$$

where $F_0' = \partial F_0/\partial J$. A solution to Eq. (11) is

$$f_1 = -F_0' \sum_{n=-\infty}^{\infty} \frac{nv_n(J)}{\omega - n\omega_s} e^{in\theta}. \quad (12)$$

Now, linearizing Eq. (7) and substituting Eq. (12) into it yields an infinite set of integral equations that determines eigenfrequencies and eigenfunctions for the collective oscillations of the bunch:

$$v_n(J) = I \sum_m m \int_0^\infty dJ_1 K_{nm}(J, J_1) \frac{F_0'(J_1) v_m(J_1)}{\omega - m\omega_s(J_1)}, \quad (13)$$

with the kernel given by

$$K_{nm} = \frac{1}{2\pi} \int_0^{2\pi} \int_0^{2\pi} d\theta d\theta_1 e^{i(m\theta_1 - n\theta)} K(J, \theta, J_1, \theta_1), \quad (14)$$

and $K(J, J_1, \theta, \theta_1) = S(x(J, \theta) - x(J_1, \theta_1))$. The integral on the right hand side of Eq. (13) defines an analytical function in the upper half plane of the complex variable ω; for $\text{Im}\omega \le 0$ the integral must be analytically continued into the lower half plane.

4 NONLINEAR THEORY

Let us assume that the instability has a threshold corresponding to a critical value of the parameter $I = I_c$ with the frequency at the threshold $\omega = \omega_c$ ($\text{Im}\omega_c = 0$). We will be interested in the analysis of the nonlinear phase of the instability in the vicinity of the threshold when the growth rate of the instability, Γ, is much smaller than ω_c, $\Gamma \ll \omega_c$. It turns out that in this case one can separate a "slow" time scale on which the amplitude evolves from "fast" oscillations with the frequency ω_c and derive nonlinear equations for the evolution of the amplitude of the instability by averaging over ω_c [7].

First, we rewrite the result of the previous section in a concise form,

$$\hat{L}(\omega, I) V_\omega = 0, \quad (15)$$

where the linear operator $\hat{L}$ represents a set of integral equations (13). A particular form of the operator $\hat{L}$ is not essential for the analysis. The frequency of the oscillations ω_c at the threshold and the corresponding eigenfunction $V_{\omega_c} \equiv u_c$ are determined by the equation

$$\hat{L}(\omega_c, I_c) u_c = 0. \quad (16)$$

We now consider a situation when I slightly exceeds the threshold, $I = I_c + \Delta I$, with $\Delta I \ll I_c$, and denote the difference $\omega - \omega_c = \Omega + i\Gamma$ (ω is now the frequency of the unstable mode above the threshold), where Γ is the growth rate, and Ω is the coherent frequency shift. Following a general prescription of nonlinear theory of oscillations [8], we will assume the following type of solution (in time representation) for $\tilde{V}$,

$$\tilde{V} = \left[A(\tau) u_c e^{-i\omega_c\tau} + \text{c.c.}\right] + \Delta V(J, \theta, \tau), \quad (17)$$

where $|Au_c| \gg |\Delta V|$. The first term in Eq. (17) describes oscillations with the eigenfunction u_c, frequency ω_c and varying amplitude $A(\tau)$, and the second term is a correction due to the deviation of the exact eigenfunction from u_c. It is important to emphasize here that $A(\tau)$ is supposed to be a slow function of time, $|\partial \ln A/\partial\tau| \ll \omega_c$.

Solving the nonlinear Vlasov equation iteratively, after cumbersome calculations that we omit because of the lack of space (see details in Ref. [7]), one can obtain an equation for the complex amplitude A. This equation contains contributions from resonances characterized by different values of the action J_n, where $n\omega_s(J_n) = \omega_c$ with n being and integer. In a typical situation, only one value of n contributes to the result due to a small synchrotron frequency spread within the bunch. Introducing scaled parameters: amplitude a, growth rate g, and time ξ according to equations $a = A\sqrt{\rho}/B_n^{5/6}e^{i\Omega\tau}$, $g = \Gamma/B_n^{1/3}$, $\xi = B_n^{1/3}\tau$, where $B_n = n^2 (\omega_s')^2 D(J_n)$, and ρ and ϕ are the absolute value and the phase of a matrix element of the kernel (see [7] for details) the equation for the amplitude takes the form

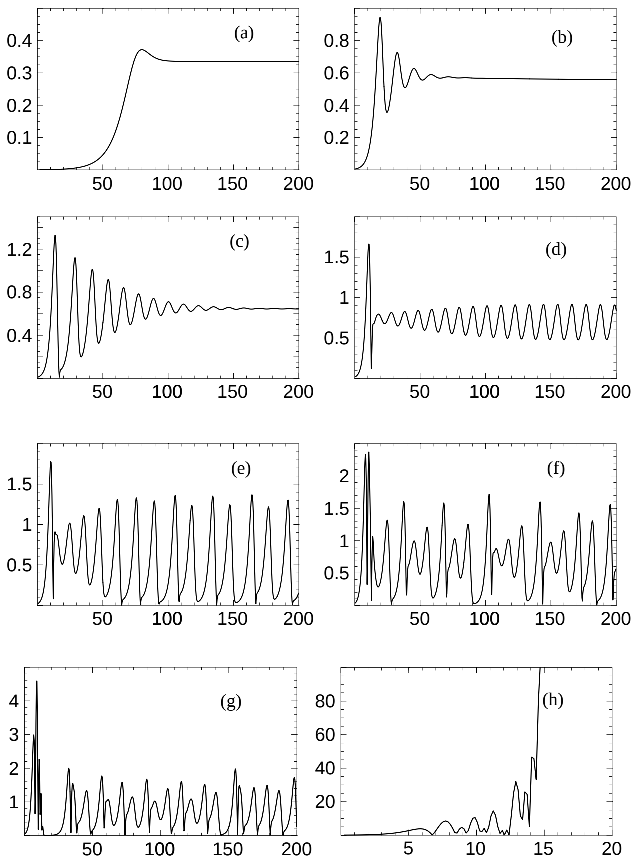

Figure 1: Plots of the absolute value of the amplitude, $|a|$, versus time ξ for $\phi = 0$. (a) – $g = 0.1$, (b) – $g = 0.3$, (c) – $g = 0.4$, (d) – $g = 0.48$, (e) – $g = 0.5$, (f) – $g = 0.6$, (g) – $g = 0.7$, (h) – $g = 0.8$.

$$\frac{\partial a}{\partial \xi} - ga = -e^{i\phi} \int_0^{\xi/2} d\zeta a(\xi - \zeta)\zeta^2$$

$$\times \int_0^{\xi-2\zeta} d\sigma a(\xi - \zeta - \sigma) a^*(\xi - 2\zeta - \sigma) e^{-\zeta^2\left(\sigma + \frac{2}{3}\zeta\right)}. \quad (18)$$

The parameter g here plays a role of dimensionless growth rate of the instability that is measured in time units related to the synchrotron damping rate. Note that Eq. (18) contains only two real parameters, g and ϕ.

5 ANALYSIS

Equation (18) admits an asymptotic solution in the form of $a = \text{const} \times \exp(i\lambda\xi)$ that corresponds to oscillations with a constant amplitude and a coherent frequency shift λ. This solution is valid in the limit $\xi \to \infty$ and exists only if $|\phi| < \pi/2$. It is given by the following formula that can be easily verified by direct substitution, $a = 18^{1/6} g^{1/2} \left(\Gamma\left(\frac{1}{3}\right)\cos\phi\right)^{-1/2} e^{-i\xi\tan\phi}$, where $\Gamma\left(\frac{1}{3}\right)$ stands for the gamma function. According to this solution, the steady state amplitude $|a|$ increases in proportion to the square root of the dimensionless growth rate, $g^{1/2}$. It turns out however, that this solution is only stable for relatively small values of the parameter g.

We have solved Eq. (18) numerically for several sets of g and ϕ. The results for $\phi = 0$ are presented in Fig. 1.

Even visual comparison of the instability signal from Ref. [1] shows a clear resemblance to some of our curves. In one case (Fig. 5 of Ref. [1]), after injection in the ring, the amplitude of signal from spectrum analyzer tuned to a sideband frequency began to grow monotonically and after some time of the order of synchrotron damping time saturated at approximately constant level. This situation is very similar to our Fig. 1a. In another case (Fig. 4 of Ref. [1]), oscillations with decreasing amplitude were observed, which can be identified with Fig. 1b or 1c. In later measurements [9], amplitude oscillations with approximately constant modulation were measured. This situation reminds our Fig. 1e.

Further work is planned to make a more definite comparison of the theory with the experiment.

6 ACKNOWLEDGMENTS

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