Analytic Models for the Broad Band Impedance

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1 Summary

In order to overcome the limitations of the broad band resonator impedance model, two different models are proposed here which have more realistic properties at both very high and very low frequencies. For each model we specify two types, which are proportional to either $\omega^{-1/2}$ or $\omega^{-3/2}$ at high frequencies. At low frequencies, the real part of the first impedance model rises quadratically or - for a special choice of the parameters - with the fourth power of frequency. In the second model it is identically zero up to a chosen "cut-off frequency", and thereby yields the very fast decrease of the loss factor with bunch length which is always found by direct solution of Maxwell's equations for lossless structures. Although not quite as simple as the broad-band resonator, both models have impedances and Green functions which may be expressed by analytic functions and depend only on few parameters. Thus they should lead to improved estimates of beam stability limitations in particle accelerators and storage rings.

2 Introduction

The stability of charged particle beams in high energy accelerators or storage rings depends critically on the shape and the electro-magnetic properties of all structures surrounding the beam. Their effect can be described either by a wake potential, which is usually a quite irregular function of the distance behind the exciting particles, or by an impedance, which is generally a rather complicated function of frequency, with a large number of peaks due to many resonant modes in all cavity-like objects. However, for the (single turn) stability of a short bunch, one needs consider only the wake potential over the length of the bunch. Such a short range wake is equivalent to a very low resolution in the frequency domain, i.e. sharp peaks of the impedance function will be smeared out.

This reduced frequency resolution of a short bunch has been the main justification for using the "broad-band resonator" model [1]: the actual impedance function is replaced by that of a single resonant circuit with a low quality factor. The whole frequency behaviour is then described with only three parameters: the resonant frequency, the shunt impedance, and the quality factor. However, it also has a number of limitations:

a) its real part falls off rather steeply as ω^{-2} at high frequencies, while the asymptotic behaviour should be only $\omega^{-1/2}$ for a single cavity with infinite side tubes[2], or $\omega^{-3/2}$ for a periodic structure[3]. This fact was only of minor importance for bunch lengths of many centimeters, and impedances due to large cavities or cross-section variations. However, for very short bunches and/or for structures with high resonant frequencies (e.g. small steps or bellows) this difference becomes more important.

b) at low frequencies, the real part increases as ω^2, while it should remain zero up to the lowest resonance for a lossless structure. For Gaussian bunches the loss factor should vanish exponentially with bunch length, but decreases only as σ^{-3} in the resonator model.

3 The Broad-band Resonator Impedance

The expression for the complex impedance of a parallel resonator is usually written

$$Z(\omega) = \frac{R}{1 + jQ(\omega/\omega_r - \omega_r/\omega)} \qquad (1)$$

where R is the shunt impedance, $f_r = \omega_r/2\pi$ the resonant frequency, and Q the quality factor. It is possible to split the RHS of Eq.(1) into two inverse linear functions by partial fraction decomposition

$$Z(\omega) = \frac{R}{jS}\left(\frac{\omega_1}{\omega - \omega_1} - \frac{\omega_2}{\omega - \omega_2}\right) \qquad (2)$$

where $S = \sqrt{4Q^2 - 1}$, and $\omega_{1,2} = \frac{\omega_r}{2Q}(j \pm S)$ are the poles of the impedance function. For $Q \geq 1/2$, S is real and hence $\omega_2 = -\omega_1^*$.

In this form, the resonator impedance permits the explicit evaluation of many integrals, often by using the residue theorem. As an example, we show here the calculation of the delta-function wake potential or "Green function": In front of the particle ($\tau < 0$), the exponential factor requires that the integration contour be closed in the lower half-plane. Since there are no poles there, the function vanishes. Behind the source ($\tau > 0$), one must integrate in the upper half-plane to find

$$\begin{aligned} G(\tau) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, Z(\omega)\exp(j\omega\tau) \\ &= \frac{R}{S}\left[\omega_1 \exp(j\omega_1\tau) - \omega_2 \exp(j\omega_2\tau)\right] \\ &= \frac{2R}{S} Re\left[\omega_1 \exp(j\omega_1\tau)\right] \end{aligned} \qquad (3)$$

where the last line holds for $Q \geq 1/2$. This expression can also be written in purely real form

$$G(\tau) = \frac{\omega_r R}{Q}\exp(-z)\left[\cos(Sz) - \frac{1}{S}\sin(Sz)\right] \qquad (4)$$

where $z = \omega_r\tau/2Q$. For $Q < 1/2$, one should replace S by $S' = \sqrt{1 - 4Q^2}$ and the circular functions by hyperbolic ones. For $Q = 1/2$, both expressions become indeterminate and must be replaced by

$$G(\tau) = 2\omega_r R\exp(-\omega_r\tau)\left[1 - \omega_r\tau\right]. \qquad (5)$$

The wake potential of a bunch can then be found by convolution of its line density λ with the Green function. For a Gaussian bunch with RMS length σ one gets

$$W(\tau) = \frac{R}{S}\exp\left(-\frac{\tau^2}{2\sigma^2}\right) Re\left[\omega_1 w(z)\right] \quad , \quad z = \frac{\omega_1\sigma}{\sqrt{2}} - \frac{j\tau}{\sigma\sqrt{2}} \qquad (6)$$

where $w(z)$ is the "complex error function"[5].

The loss factor can be calculated from the integrals

$$k(\sigma) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, Z(\omega)h(\omega) \;=\; \int_0^{\infty} d\tau\, G(\tau)s(\tau) \qquad (7)$$

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where $h(\omega) = \tilde{\lambda}^2(\omega)$ is the spectral power density, and $s(\tau)$ the auto-correlation function of the line density. For a Gaussian bunch in a resonator, the loss factor becomes

$$k(\sigma) = \frac{R}{S} Re\left[\omega_1 w(\omega_1 \sigma)\right]. \tag{8}$$

Asymptotically, it depends on both R and Q separately, and not only on R/Q as in the transverse case [4], and thus the resonator parameters cannot be determined uniquely.

4 The First Improved Impedance Model

The real and imaginary parts of a complex impedance are related by a Hilbert transformation. Therefore they are the cosine and sine transforms of a single, real "Green" function which vanishes for $\tau < 0$. Not many functions are known to have explicit analytic sine and cosine transforms, and if one furthermore looks for a particular asymptotic behavior, the choice is restricted to very few.

Impedance Model 1A

The first candidate[6], with an asymptotic frequency dependence of the cosine transform approaching $\omega^{-3/2}$, is the complementary error function of the square root of $y = \omega_1\tau$. For $y > 0$

$$G(y) = \mathrm{erfc}(\sqrt{y}) \tag{9}$$

with the Fourier transforms

$$\begin{aligned}\int_0^\infty dy\, G(y)\cos(xy) &= \frac{1}{2u}\sqrt{\frac{2}{u+1}} \\ \int_0^\infty dy\, G(y)\sin(xy) &= \frac{1}{x} - \frac{1}{2u}\sqrt{\frac{2}{u-1}}\end{aligned} \tag{10}$$

where

$$x = \frac{\omega}{\omega_1}, \quad u = \sqrt{x^2+1}. \tag{11}$$

Any realistic impedance must have a vanishing real part at zero frequency since a uniform, coasting beam does not lose energy to the vacuum chamber walls. One can thus add an impedance which just cancels the real part at zero frequency, but falls off faster at high frequencies. We may take e.g. resonator impedance, since its real part decreases asymptotically as ω^{-2}.

The full expression for the impedance and Green function are shown in Table I. The frequencies ω_1, ω_2, and the 'shunt impedance' R form three parameters which may be chosen to fit the impedance of any particular structure. For very high frequencies, the asymptotic behaviour of this impedance model is given by

$$ReZ(x) \propto x^{-3/2}, \quad ImZ(x) \propto \frac{2-\alpha}{x} \tag{12}$$

while at low frequencies one obtains the approximations

$$ReZ(x) \propto \left(\frac{1}{\alpha^2} - \frac{5}{8}\right)x^2, \quad ImZ(x) \propto \left(\frac{1}{\alpha} - \frac{3}{4}\right)x \tag{13}$$

In order to have a positive definite real part of the impedance, one has to take $\alpha^2 \leq 8/5$, which also yields a positive imaginary part (inductive) at low frequencies. It becomes negative (capacitive) at high frequencies just as for the resonator impedance. For $\alpha^2 = 8/5$, the real impedance increases as ω^4 at low frequencies a shown in Fig.1. This choice results in a faster decrease of the loss factor with bunch length ($\propto \sigma^{-5}$).

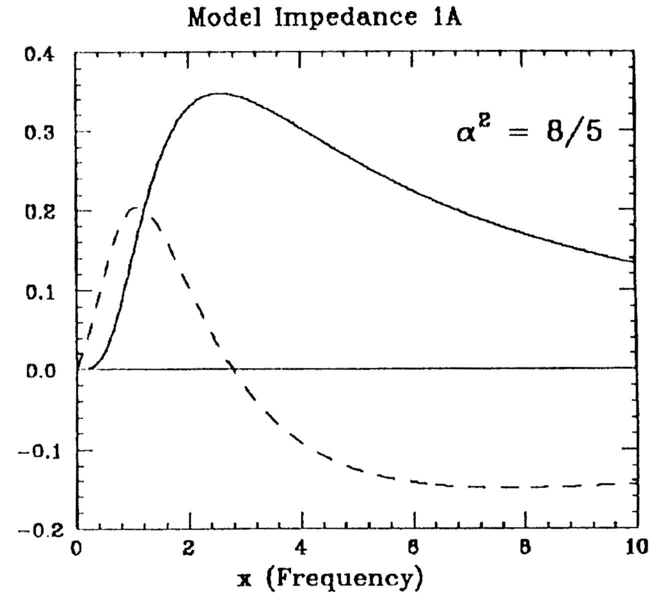

Figure 1: Real and Imaginary Parts of Impedance

Impedance Model 1B

For an inverse square-root dependence of the real part of the impedance at high frequencies, a suitable pair of Fourier transforms can be found with the Green function

$$G(y) = \exp(-y)/\sqrt{y} \tag{14}$$

With $u = \sqrt{x^2+1}$ as before, the transforms can be written

$$\begin{aligned}\int_0^\infty dy\, G(y)\cos(xy) &= \sqrt{\frac{\pi}{2}}\frac{\sqrt{u+1}}{u} \\ \int_0^\infty dy\, G(y)\sin(xy) &= \sqrt{\frac{\pi}{2}}\frac{\sqrt{u-1}}{u}\end{aligned} \tag{15}$$

Again one has to add a term to make the real part of the impedance vanish at zero frequency. With $x = \omega_1/\omega$ and $\alpha = \omega_2/\omega_1$, this yields the complex impedance and the corresponding Green function shown in Table I. At low frequencies, the normalized impedance can be approximated by the expressions

$$ReZ(x) \propto \left(\frac{1}{\alpha^2} - \frac{3}{8}\right)x^2, \quad ImZ(x) \propto \left(\frac{1}{\alpha} - \frac{1}{2}\right)x \tag{16}$$

In order to guarantee a positive real part, one now has to take $\alpha^2 \leq 8/3$. The loss factor for this impedance model can be calculated from the auto-correlation function. For a Gaussian bunch, one obtains integrals which can be expressed explicitly by the "parabolic cylinder function" $D_{-1/2}$. Unfortunately, these functions are not standard computer library routines. However, we could derive a relation to modified Bessel functions of order 1/4 which is not given in standard textbooks:

$$D_{-1/2}(x) = \sqrt{\frac{x}{2\pi}}K_{1/4}\left(\frac{x^2}{4}\right) \tag{17}$$

This makes numerical evaluation easier, and simplifies the derivation of approximations for the loss factor [8].

5 Second Improved Impedance Model

For any impedance whose real part increases as ω^{2n} at low frequencies, the loss factor depends asymptotically as $\sigma^{-(2n+1)}$ on

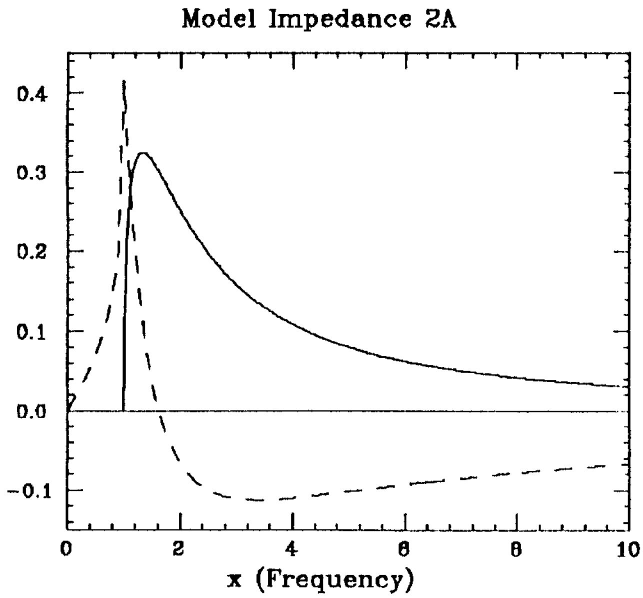

Figure 2: Real and Imaginary Parts of Impedance

the bunch length. Thus no power law can lead to the fast, exponential fall-off found for Gaussian bunches with computer codes integrating Maxwell's equations [7]. In order to obtain such a decrease, it is necessary that the real part of the impedance remain zero up to a "cut-off" frequency ω_1. We have constructed two impedance models with this property, whose real parts decrease as either $\omega^{-3/2}$ or $\omega^{-1/2}$ at high frequencies.

Impedance Model 2A

In this model, the real part of the impedance vanishes for $|x| < 1$, while for $|x| > 1$ it is proportional to

$$ReZ(x) = \frac{\sqrt{|x| - 1}}{x^2} \qquad (18)$$

where $x = \omega/\omega_1$. The real part has a maximum for $x = 4/3$, and decreases with $\omega^{-3/2}$ at high frequencies. The corresponding imaginary part can again be obtained by Hilbert transformation, and is listed in Table I. For low frequencies, it is proportional to $2x/3$, and has a sharp peak at x=1. For high frequencies, it tends to $-1/x$. The corresponding normalized Green function $\bar{G} = G/\omega_1 R$ is given by

$$G(y) = \frac{2}{\pi}\int_1^\infty dx\, \frac{\sqrt{x-1}}{x^2}\cos(xy) \qquad (19)$$

where $y = \omega_1 \tau$, leading to integrals over Fresnel functions. For computer evaluation, this is of no advantage over direct numerical integration, in particular since the integrand converges reasonably well. Also the normalized loss factor

$$k(\alpha) = \frac{1}{\pi}\int_1^\infty dx\, \frac{\sqrt{x-1}}{x^2}\exp(-\alpha^2 x^2) \qquad (20)$$

must be obtained by numerical integration. The result shows the asymptotically exponential decrease typical for lossless structures.

Impedance Model 2B

One can construct a similar model for an impedance whose real part decreases asymptotically as the inverse square root of frequency. We simply replace x^2 by $|x|$ in Eq.18. This function has a maximum value of $Z = 1/2$ for $x = 2$. The corresponding imaginary part is again found with the Hilbert transformation, and shown in Table I. At low frequencies, the imaginary part is inductive and peaks sharply at $x = 1$. It decreases rapidly, changes sign, and finally approaches zero logarithmically at high frequencies. The corresponding normalized Green function can be evaluated in terms of Fresnel integrals S(x) and C(x) and is given in Table I. However, the loss factor of a Gaussian beam must again be obtained by numerical integration.

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	$1A$	$1B$	$\|x\|$	$2A$	$2B$
$Z_R(x)$	$\frac{1}{u}\sqrt{\frac{2}{u+1}} - \frac{\alpha^2}{x^2+\alpha^2}$	$\frac{\sqrt{u+1}}{u\sqrt{2}} - \frac{\alpha^2}{x^2+\alpha^2}$	< 1	0	0
			> 1	$\frac{\sqrt{\|x\|-1}}{x^2}$	$\frac{\sqrt{\|x\|-1}}{\|x\|}$
$Z_I(x)$	$\frac{1}{u}\sqrt{\frac{2}{u-1}} - \frac{\alpha x}{x^2+\alpha^2} - \frac{2}{x}$	$\frac{\alpha x}{x^2+\alpha^2} - \frac{\sqrt{u-1}}{u\sqrt{2}}$	< 1	$\frac{\sqrt{1+x} - \sqrt{1-x} - x}{x^2}$	$\frac{2 - \sqrt{x+1} - \sqrt{1-x}}{x}$
			> 1	$\frac{\sqrt{1+x} - x}{x^2}$	$\frac{2 - \sqrt{1+x}}{x}$
$G(y)$	$\left[2\mathrm{erfc}(\sqrt{y}) - \alpha e^{-\alpha y}\right]$	$\frac{e^{-y}}{\sqrt{\pi y}} - \alpha e^{-\alpha y}$		$\frac{2}{\pi}\int_1^\infty dx \frac{\sqrt{x-1}}{x^2}\cos(xy)$	$2\left[\frac{\cos y - \sin y}{\sqrt{2\pi y}} + C(z) + S(z) - 1\right]$

Impedances and Green functions; $x = \omega/\omega_1$, $u = \sqrt{x^2+1}$, $y = \omega_1\tau$, $z = \sqrt{2y/\pi}$