# THREE-DIMENSIONAL WAKE FIELD ANALYSIS BY BOUNDARY ELEMENT METHOD

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## Abstract

A computer code HERTPIA was developed for the calculation of electromagnetic wake fields excited by charged particles travelling through arbitrarily shaped accelerating cavities. This code solves transient wave problems for a Hertz vector. The numerical analysis is based on the boundary element method. This program is validated by comparing its results with analytical solutions in a pill-box cavity.

## Introduction

Beam loading effects strongly limit the performance of linear accelerators and storage rings. Electromagnetic interactions between charged particles and surroundings such as beam ducts and cavities produce wake fields to cause beam instabilities. Wake field analysis, therefore, is indispensable for accelerator design. A three dimentional analysis is required to treat non-axi-symmetric beam behaviors and structures such as beam ducts, beam duct joints, RF couplers, etc.

Three-dimensional analysis code have already been developed; WELL [1] and 3D-BCI [2]. However, the code WELL is applicable only to structures with surfaces parallel and perpendicular to the beam. The code 3D-BCI can treat arbitrarily shaped structures, but it may require long CPU time as well as complicated mesh generation in space.

We developed the computer code HERTPIA (Hertz-vector-based particle-inducted-field analysis program) in which a transient wave equation of the electric Hertz vector is solved by the boundary element method. In HERTPIA, the mesh is generated only on the boundary surface and the CPU time is reduced because of permitting rough mesh compositions and large time steps.

## Basic Equation

In a vacuum, electromagnetic fields are described by the Hertz vector $\Pi$ as follows,

$$E = \nabla(\mathrm{div}\,\Pi) - \frac{\partial^2 \Pi}{\partial T^2}, \tag{1}$$

$$H = \frac{1}{Z_0}\frac{\partial}{\partial T}\,\mathrm{rot}\,\Pi, \tag{2}$$

$$\left(\nabla^2 - \frac{\partial^2}{\partial T^2}\right)\Pi = P, \tag{3}$$

$$P = -Z_0\int_{-\infty}^{T} J(T')\,dT', \tag{4}$$

where
T=ct
c:light velocity
t:time,
E:electric field vector,
J:vector of beam current density,
Zo:characteristic impedance in a vacuum.

The transient wave equation (3) can be rewritten in the following integral form,

$$\beta_i \Pi(r_i, T) = \left[\int_\Gamma \left(\frac{1}{R}\frac{\partial \Pi(r,T')}{\partial n} + \frac{R'}{R^2}\frac{\partial \Pi(r,T')}{\partial T'} + \frac{R'}{R^3}\Pi(r,T')\right)d\Gamma \int_\Omega \frac{P(r,T')}{R}\,d\Omega\right]_{T'=T-R}, \tag{5}$$

where

$$R = |r - r_i|, \quad R' = (r - r_i)\cdot n,$$

$\beta_i$:a soild angle viewed from the point $r_i$ to the computational domain $\Omega$,
$\partial/\partial n$:differential operator in the direction normal to the boundary surface $\Gamma$,
n :outside-viewing normal unit vector on the boundary surface $\Gamma$.

Equation (5) is the basic equation in the boundary element method.

Based on the Hertz vector, the boundary conditions on perfectly conducting walls are given in simple forms,

$$\Pi = (n\cdot\Pi)\,n, \quad \frac{\partial}{\partial n}(n\cdot\Pi) = 0. \tag{6}$$

The computational domain $\Omega$ is confined in a finite space, which means that the boundary surface $\Gamma$ includes some open boundaries. For the boundary conditions imposed on the open boundaries, the approximate techniques used in the code TBCI [3] and DBCI [4] are employed here to get the following open boundary conditions,

$$\frac{\partial \Pi}{\partial n} = 2\frac{\partial \Pi_b}{\partial T} - \frac{\partial \Pi}{\partial T} \quad (\text{at beam entrance}),$$
$$\frac{\partial \Pi}{\partial n} = -\frac{\partial \Pi}{\partial T} \quad (\text{at the others}), \tag{7}$$

where

$\Pi_b$:beam induced field in a free space.

Suppose that an exciting charge enters an empty cavity at z=0 and leaves at z=L with constant velocity v parallel to the z-axis, and a test charge also moves at the same velocity at transverse position (x,y) and at longitudinal position s behind the exciting charge. The wake potential defined as the momentum kick experienced by the test charge is expressed by the following equation,

$$V(s,x,y) = \int_0^L [E + (v\times B)]_{T_0}\,dz$$
$$= \left[\left(\nabla\Pi_z - \frac{\partial \Pi}{\partial T}\right)\right]_{z=0}^{z=L} + \int_0^L [\nabla(\nabla_\perp \Pi)]_{T_0}\,dz, \tag{8}$$

where

$$\nabla_\perp = \frac{\partial}{\partial x} + \frac{\partial}{\partial y},$$

$$T_0 = (s+z)/\beta,$$

$$\beta = v/c,$$

$\Pi_z$ :z-component of Hertz vector $\Pi$.

## Discretization

The integral equation (5) is solved by the boundary element method. The equation can be written in a discretized form as

$$\beta_i \Pi(r_i, T) = \sum_{j=1}^{N_0} \Big[ A_{ij} \frac{\partial \Pi(r_j, T')}{\partial n} + B_{ij} \frac{\partial \Pi(r_j, T')}{\partial T'} + C_{ij} \Pi(r_j, T') \Big]_{T'=T-R_{ij}} - \sum_{\ell=1}^{M_0} D_{i\ell} [P_\ell(T')]_{T'=T-R_{i\ell}}, \quad (9)$$

where

No: total number of nodes on boundary $\Gamma$,

Mo: total number of nodes constituting local domains including sources,

$$\beta_i = \sum_{j=1}^{N_0} B_{ij},$$

$$A_{ij} = L_{ij}\left(\frac{1}{R}\right), \qquad B_{ij} = L_{ij}\left(\frac{R'}{R^2}\right),$$

$$C_{ij} = L_{ij}\left(\frac{R'}{R^3}\right), \qquad D_{i\ell} = \overset{*}{L}_{i\ell}\left(\frac{1}{R}\right),$$

$$R_{ij} = |r_i - r_j|, \qquad R_{i\ell} = |r_i - r_\ell|,$$

$P_\ell$ : source vector P in spatial element 1,

$$L_{ij}(x) = \sum_{n=1}^{N} \sum_{k=1}^{m(n)} \delta_{j,J(\Gamma_n,k)} \int_{\Gamma_n} x N_k^{(2)} d\Gamma,$$

$\Gamma_n$ : n-th boundary element,

$J(\Gamma_n, k)$ :k-th node number around $\Gamma_n$,

N : number of boundary elements,

m(n): number of nodes in $\Gamma_n$,

$N_k^{(2)}$ : k-th two dimensional shape function,

$$\overset{*}{L}_{i\ell}(x) = \sum_{n=1}^{\widetilde{N}} \sum_{k=1}^{\widetilde{m}(n)} \delta_{\ell, J(\Omega_n,k)} \int_{\Omega_n} x N_k^{(3)} d\Omega,$$

$\Omega_n$ : n-th spatial element including source,

$J(\Omega_n, k)$ :k-th node number around $\Omega_n$,

$\widetilde{N}$ : number of spatial elements,

$\widetilde{m}(n)$: number of nodes in $\Omega_n$,

$N_k^{(3)}$: k-th three dimensional shape function,

$$\delta_{j,J} = \begin{cases} 1 & (j = J) \\ 0 & (j \neq J) \end{cases}.$$

Solving the discretized equations (9) with boundary condition equations (6) (7), we get the sets of Hertz vectors $\Pi$ and their normal derivatives $\partial\Pi/\partial n$ at all boundary nodes, and finally we can obtain fields at arbitrary points in the space $\Omega$ or on the boundary surface $\Gamma$.

We use the following equations to get Hertz vector fields $\Pi$ and $\partial\Pi/\partial n$ on the all boundary nodes.

(1) On the perfectly conducting walls

$$\Pi_i = \Phi_i n,$$

$$\frac{\partial \Pi_i}{\partial n} = \frac{1}{A_{ii}} \Big[ (\beta_i - C_{ii}) \Phi_i n - B_{ii} \frac{\partial \Phi_i}{\partial T} n - Q_i \Big]$$

where

$$Q_i = \sum_{j \neq i} \Big[ A_{ij} \frac{\partial \Pi_{j(T')}}{\partial n} + B_{ij} \frac{\partial \Pi_{j(T')}}{\partial T'} + C_{ij} \Pi_{j(T')} \Big]_{T'=T-R_{ij}} - \sum_{\ell} D_{i\ell} [P_{\ell(T')}]_{T'=T-R_{i\ell}},$$

$\Pi_j$: Hertz vector value at node j,

and $\Phi_i$ satisfies the following equation,

$$B_{ii} \frac{\partial \Phi_i}{\partial T} + (C_{ii} - \beta_i) \Phi_i + n \cdot Q_i = 0.$$

(2) On the open boundaries

$$\frac{\partial \Pi_i}{\partial n} = 2 \frac{\partial \Pi_{bi}}{\partial T} - \frac{\partial \Pi_i}{\partial T} = \frac{1}{A_{ii}} \Big[ (\beta_i - C_{ii}) \Pi_i - B_{ii} \frac{\partial \Pi_i}{\partial T} - Q_i \Big],$$

where $\Pi_i$ satisfies the following equation,

$$(A_{ii} - B_{ii}) \frac{\partial \Pi_i}{\partial T} + (\beta_i - C_{ii}) \Pi_i = Q_i + 2A_{ii} \frac{\partial \Pi_{bi}}{\partial T},$$

$\Pi_{bi}$ : beam induced field at the i-th node in a free space.

If the open boundary is not a beam entrance, the beam induced field $\Pi_{bi}$ vanishes.

(3) On the cross lines between the perfectly conducting walls and the open boudaries

Two kinds of boundary conditions are given at the nodes on the cross lines between the perfectly conducting walls and the open boundaries. Double nodes are conveniently defined. The node numbers i and 1 are assigned to the double nodes with the i-th node on the perfectly conducting wall and 1-th node on the open boundary.

The boundary fields are obained from the following equations,

$$\Pi_i = \Pi_\ell = \Phi n_i,$$

$$\frac{\partial \Pi_i}{\partial n_i} = \frac{1}{A_{ii}} \Big( \Gamma_{i\ell} n_i - 2A_{i\ell} \frac{\partial \Pi_{b\ell}}{\partial T} - Q_{i\ell} \Big),$$

$$\frac{\partial \Pi_\ell}{\partial n_\ell} = 2 \frac{\partial \Pi_{b\ell}}{\partial T} + \frac{n_i}{\overline{B}_{i\ell}} (\overline{C}_{i\ell} \Phi + \Gamma_{i\ell}),$$

where

$$Q_{i\ell} = \sum_{j \neq i,\ell} \Big[ A_{ij} \frac{\partial \Pi_{j(T')}}{\partial n} + B_{ij} \frac{\partial \Pi_{j(T')}}{\partial T'} + C_{ij} \Pi_{j(T')} \Big]_{T'=T-R_{ij}} - \sum_{\ell} D_{i\ell} [P_{\ell(T')}]_{T'=T-R_{i\ell}}.$$

$$\overline{B}_{i\ell} = B_{ii} + B_{i\ell} - A_{i\ell},$$

$$\overline{C}_{i\ell} = C_{ii} + C_{i\ell} - \beta_i,$$

$$\Gamma_{i\ell} = \Big( 2A_{i\ell} \frac{\partial \Pi_{b\ell}}{\partial T} + Q_{i\ell} \Big) \cdot n_i,$$

and $\Phi$ satisfies the following equation,

$$\overline{B}_{i\ell} \frac{\partial \Phi}{\partial T} + \overline{C}_{i\ell} \Phi + \Gamma_{i\ell} = 0.$$

## Comparison with analytical solution

The code HERTPIA has been verified in comparison with exact analytical solutions for closed cylindrical cavities (pill-box cavities). The solutions are obtained by the modal analysis in the frequency domain.

Suppose an exciting charge with infinitesimal radius and with Gaussian density distribution along the beam axis, travells along the central axis of the pill-box cavity, the longitudinal wake potential $V_z(s)$ is given by

$$V_z(s) = -\frac{\sqrt{2}\,Z_0\,\sigma\,\hat{I}}{\pi L}\sum_{n=1}^{\infty}\sum_{p=-\infty}^{\infty}\left(\frac{1-(-)^p\cos(\nu_{np}L)}{j_n^2\,J_1^2(j_n)}\right)G_{np}, \tag{10}$$

with

$$G_{np} = \frac{\sqrt{2}}{\sigma}\int_0^{\infty} e^{-\frac{(s-\xi)^2}{2\sigma^2}}\cos(\nu_{np}\xi)\,d\xi \tag{11.a}$$

$$= 2\sqrt{\pi}\,e^{-\frac{\sigma^2\nu_{np}^2}{2}}\cos(\nu_{np}s) - e^{-\frac{s^2}{2\sigma^2}}\,I_m[Z_p(\theta)], \tag{11.b}$$

$$\theta = \frac{\sigma}{\sqrt{2}}\left(j\frac{s}{\sigma^2} - \nu_{np}\right),$$

$$\nu_{np} = \sqrt{\left(\frac{j_n}{R_0}\right)^2 + \left(\frac{\pi p}{L}\right)^2},$$

where

- $R_o$: radius of cavity,
- $L$ : length of cavity,
- $J_1$: 1-st order Bessel function,
- $j_n$: n-th zero of 0-th order Bessel function,
- $\sigma$ : standard deviation of beam distribution,
- $\hat{I}$ : peak current,
- Im[ ]: imaginary part of value inside [ ].

The plasma dispersion function $Z_p(\theta)$ is defined as follows,

$$Z_p(\theta) = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}\frac{e^{-u^2}}{u-\theta}\,du. \tag{12}$$

By using Eq.(12), the calculation of the right hand side of Eq.(11.a) converges rapidly with less computational errors.

Numerical results obtained by HERTPIA was compared with the exact analytical solutions given by Eq.(10). As an example, we set L=10cm, Ro=5cm, $\sigma$=2.5cm, and $\hat{I}$=1A. It is verified that the value Vz(s) converges in sufficient accuracy with the truncation mode number $n \gtrsim 30$, $|P| \gtrsim 80$ in Eq.(10). Figure 1 shows the boundary mesh composition in the HERTPIA calculation. The numbers of divisions are 6 in the radial, 16 in the circumferential, and 10 in the axial directions respectively, and the total numbers of elements and nodes are 336 and 370, respectively. Figure 2 compares the results by HERTPIA and the analytical solutions with respect to the longitudinal wake potential. The Gaussian curve indicates the line charge density distribution of the exciting charge. The longitudinal wake potential is shown by the two curves. The soild line indicates the HERTPIA solution, and the broken line indicates the exact analytical solution, which is obtained with the truncation mode number n=50 and $|p| \leq 100$ The two curves almost coincide with each other within the electron bunch ( $|s| \leq 4\sigma$ ), while the difference between them at the back of the bunch is less than 2.5% of the maximum peak-to-peak value.

The CPU time required in the calculation of the HERTPIA solution was about 5 minutes in the HITAC M-200H computer with time step $c\Delta t$=0.5cm.

## Conclusions

We have developed the computer code HERTPIA which is capable of calculating three-dimentional wake fields. The formulation is based on the Hertz vector, and the computational technique is based on the boundary element mrthod. This program was validated by comparing its results with the exact analytical solution in a pill-box cavity model. The characteristics of HERTPIA are simple mesh generation, sufficient accuracy with short CPU time and easy calculation of fields at arbitrary points.


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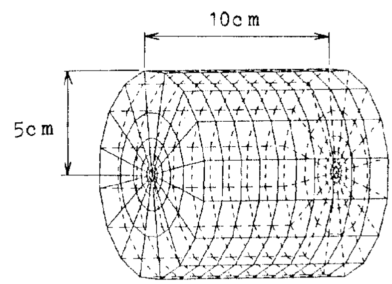


Fig.1. Boundary mesh composition

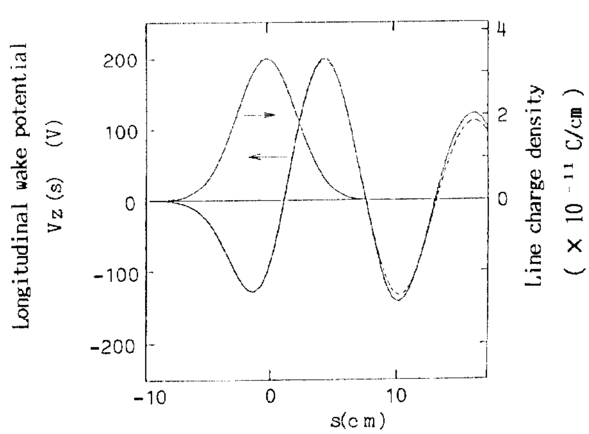


Fig.2 Longitudinal wake potential by Gaussian beam

( ——— HERTPIA solution
------- Analytical solution )