Computer Calculation of the Longitudinal Impedance of Cylindrically Symmetric Structures and its Application to the SPS

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Introduction

The coupling impedance is one of the major parameters which determine the performance of an accelerator. It is in general sufficient to quote the longitudinal value of the impedance from which the transverse value can easily be derived. In a machine like the SPS many of the vacuum chamber volumes can be conveniently approximated by cylinders. It is therefore justified to concentrate the calculations on these kinds of structures. In principle, the coupling impedance of these structures can be obtained from existing programs which give the response to the passage of a bunch of charges in the time domain [1]. However, it is difficult to probe the high frequency domain in this way and a new program (CISLIM) has been written to directly evaluate in the frequency domain the longitudinal impedance of an arbitrary sequence of cylindrical cross-section variations. The results of the program are presented for bellows , as well as for the various cylindrical structures which appear in the SPS. The computed SPS impedance is then compared with the impedance values derived from beam measurements.

Theoretical Development of the Equations

The recipe of the computation has been given by H. Hereward in 1975 [2]. The detailed calculation can be found in [3].

Smooth wall case

A charge is travelling down the centre line of a cylinder according to a $e^{j\omega t-\gamma s}$ propagation law. Symmetry imposes solutions which belong to the TM wave family. Hence only the components H_ϕ, E_r and E_s will be different from zero. Furthermore, all the derivatives with respect of ϕ will be zero due to the symmetry in ϕ. From Maxwell's equations the following Bessel differential equation can be derived using the substitution : $k = \omega/c$, where ω is the angular frequency and c the speed of light.

$$\frac{\delta^2 H_\phi}{\delta r^2(\gamma^2+k^2)} + \frac{1}{r\sqrt{(\gamma^2+k^2)}}\,\frac{\delta H_\phi}{\delta r\sqrt{(\gamma^2+k^2)}} + H_\phi\left(1-\frac{1}{r^2(\gamma^2+k^2)}\right)=0$$

The Bessel function of the first kind is chosen since it ensures finite values in the centre of the pipe.

$$H_\phi = AJ_1[r\sqrt{(\gamma^2 + k^2)}]$$

$$E_s = (A/j\omega\epsilon_o)\sqrt{(\gamma^2 + k^2)}J_o[r/\gamma^2 + k^2)]$$

$$E_r = (\gamma A/j\omega\epsilon_o)J_1[r(\gamma^2 + k^2)]$$

The integration constant A can be found from the boundary conditions. When the material has infinite conductivity $E_s = 0$ at the inner pipe surface, Hence γ will take values such that $J_o[a!(\gamma^2 + k^2)]=0$. So that a $\sqrt{(\gamma^2{}_m + k^2)} = z_m$ where z_m is the mth zero of J_o. The pipe radius a is used as a normalisation coefficient. This leads to the new geometric variables p = r/a, u = s/a, the normalized wave number k_n = ak and the normal propagation constant $\Gamma_m = a\gamma$. The solutions for the three field components can be written for every possible Γ_m:

$$B_{\phi m} = j\mu A_m J_1(pz_m)e^{j\omega t-\Gamma_m u}$$

$$E_{sm} = \sqrt{(\mu/\epsilon_0)}(z_m/k_n)A_m J_0(pz_m)e^{j\omega t-\Gamma_m u}$$

$$E_{rm} = \sqrt{(\mu/\epsilon_0)}(\Gamma_m/k_n)A_m J_1(pz_m)e^{j\omega t-\Gamma_m u}$$

Cylindrical pipe with discontinuity

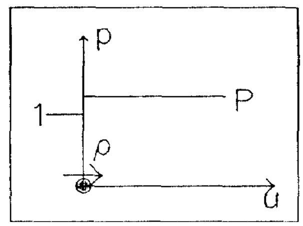

Fig. 1

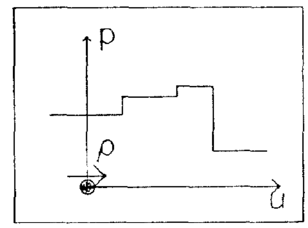

Fig. 2

The system is shown in figure 1. In the general case a wave F travels forward and a wave B backward with respect to the discontinuity. The origin of these waves can be understood from the requirement of field continuity. Indeed, the space charge is perfectly continuous at the step for 0 < p < 1 but has to be balanced by the F-wave for 1 < p < P. The link between space charge and field is given by Ampère's or Gauss' law:

$$2\pi r E_{r\ sc} = \rho/\epsilon_o$$

$E_{r\ sc}$ is the space charge field. For 1<p<P, $E_r + E_{r\ sc} = 0$ at the right hand side of the step. The continuity equations between the F-wave and B-wave can be written. The integration constants A_m are replaced by F_m and B_m respectively. This yields the following set of equations

$$\rho c k_n/2\pi a p = -\Sigma_m \Gamma_{Pm} J_1 F_m \qquad (1 < p < P)$$

$$\Sigma_m \Gamma_m J_1 B_m = -\Sigma_m \Gamma_{Pm} J_1 F_m \qquad (p < 1)$$

The factors zp_m and Γp_m are respectively the zero's of the Bessel functions and the corresponding propagation constant in a pipe with normal radius p.

Note that the equations contain an infinite sum in m. In fact it very much resembles a Fourier analysis if only the J_o and J_1 were cos and sin! The Bessel functions have orthogonal properties similar to the cos and sin functions and this can be used to derive the following matrix equation which can be solved for B and F:

$$B - T_1.F = 0$$

$$T_2.B + F = C$$

The matrices T_1 and T_2 are determined by the geometry and the column matrix C by the properties of the passing charge.

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Extension to many discontinuities

The configuration is shown in figure 2.

Waves are generated at the two steps when an electric charge passes through the pipe. At each step the continuity equations can be written down as was done for a single discontinuity. Proceeding along the same lines the following set of matrix equations is obtained:

$$B_1 - T_1.F_1 - T_1.AT.B_2 = 0$$

$$T_2.B_1 + F_1 - AT.B_2 = -Ce^{jk_n u_1}/\beta$$

$$- AT.F_1 + B_2 + T_2.F_2 = Ce^{jk_n u_2}/\beta$$

$$- AT.T_1.F_1 - T_1.B_2 + F_2 = 0$$

The matrices $\mathbf{T_1}$ and $\mathbf{T_2}$ are exactly the same as before. The new matrix AT is a diagonal matrix. The waves B_1 and F_1 participate directly in the equation while B_2 first propagates from u_2 to u_1. This feature is taken care of by AT. Its diagonal elements are:

$$AT(m,m) = \exp[-\ Q_{p_m}(u_2 - u_1)]$$

The exponential factor of the C-term on the right comes about from the same reasons. The previous calculation makes it possible to compute a single bellows convolution or an open-ended cavity. However, bellows are very rarely made up of one convolution only and important vacuum chamber sections in the SPS are much more complicated than that. The method can be generalized for many steps. The matrices T_1, T_2 and C are determined at each discontinuity.

The calculation of the longitudinal impedance

The longitudinal impedance is equal to the longitudinal electric field E_s integrated along the centre of the structure divided by the passing current. In other words

$$Z_{\|}(\omega) = (a/\rho\beta c)\int E_s(p=0, t=ua/\beta c, u)du \qquad (12)$$

The integral extends from $-\infty$ to $+\infty$.
The waves exist in 4 types B_1, F_1, B_2, F_2, each with a given number of spatial modes. The integral can be calculated for each wave, i.e. for B_1:

$$Z_{\| B_1 m} = B_{1m} z_{1m}/(\beta k_n)(jk_n + \Gamma_{1m})e^{(jk_n + \Gamma_{1m})u}$$

For the 'internal' integration limits, i.e. the ones determined by the longitudinal position of the discontinuity just replace u in (1) by the corresponding normalised longitudinal step coordinate. The value of the integrand at the 'external' integration limits is taken to be zero.

Results obtained for bellows

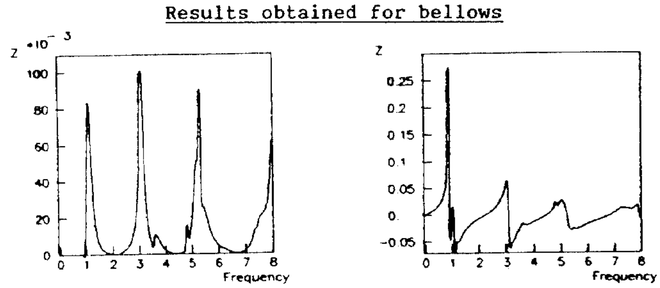

Fig. 3 Impedance of bellows

The result of an impedance analysis of a bellows structure is shown in fig. 3. This impedance is compatible with the response of the structure to the excitation of a short bunch obtained with TBCI[1]. However, impedance analysis contains more information than the time domain analysis. Note the very typical repeating resonant peaks at higher frequencies. These features recall the properties of open or short circuited loss-less electrical lines. Such an electrical line can be identified in the bellows structure. The wavelength of the resonant frequency is equal to 4 times the electrical length of the short circuited parallel line formed by the side walls of an undulation. The short circuit is simply the pipe material at radius P. The mechanical length of this line is P-1. The electrical length = P-1 + G/2. Many simulation results obtained with CISLIM have confirmed the model. The outputs clearly indicate the resonator is heavily damped. In other words the resonating structure loses energy via the waves in the end tubes. It is possible to calculate analytically the damping of the resonator [3].

Calculation of the SPS Impedance from Vacuum Chamber Geometry

The SPS vacuum system contains many bellows but their contribution to the impedance is very small. In fact, the vacuum pump-port chamber at every magnet unit turns out to be one of the most contributing elements. Moreover, it is a more complicated structure than a bellows and several resonances can be identified (fig. 4). The lowest resonance occurs at f = 1.35 GHz with an extremely high quality factor. The resonant frequency corresponds to the frequency of the lowest propagation mode in the largest part of the cavity. In practice, these resonances are damped by small resistive cylinders installed in the enlarged chamber.

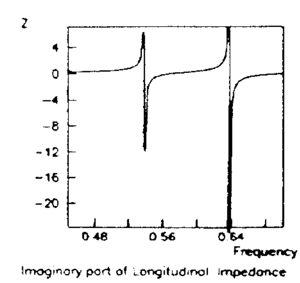

Fig. 4 Impedance of Dumping Port

Computation of the SPS Impedance from the hardware

The following table gives an overview of the results of computer calculation of $Z_{\|}$ of the various types of vacuum chamber that exist in the SPS.

The values found under the column heading Z/n are calculated with the program CISLIM while $Z_\perp$ has been derived using the classical formula

$$Z_\perp = (2R/b^2)(Z/n)$$

where b is the radius of the continuous beam tube. The sign is relative in the sense that for a flat chamber the coherent horizontal tune shift caused by the wall impedance has the opposite sign of the tune shift in the vertical plane.

The main contributions to the impedance come from vacuum port chambers (part of 'Special chambers'), the accelerating cavities and to a lesser extent from the special magnet chambers. The lowest resonances in these last two structures are well within the frequency spectrum of SPS bunches. The Z/n quoted in table I for these elements is derived from the low frequency inductance. It can be shown that this is correct for the SPS based on the calculation of an effective impedance [3].

Table 1: SPS impedance computed from hardware

ELEMENT	Z/n Ω	$Z_\perp$ Hor MΩ/m	$Z_\perp$ Ver MΩ/m
ISOLATED BELLOWS	0.1214	0.0364	0.0364
TRANSITIONS	0.1261	0.045	0.045
SPECIAL CHAMBERS	2.41	-5.4829	8.7478
SPECIAL MAGNETS	0.552	-1.337	2.136
DIRECTIONAL COUPLERS	0.0172	0.0376	0.0376
CAVITIES	2.9183	1.5136	1.5136
TOTAL	**6.2**	**−5.2**	**12.5**

From what precedes it can be concluded that the SPS impedance can be approximated by a resonator at 1.35 GHz (the lowest resonant frequency of the vacuum port chamber) and some low quality factor determined by the damping resistors in the same chamber.

Measurement of the SPS impedance with beam

In the following experimental results are reported concerning the inductive or imaginary part of Z/n, the inductive part of $Z_\perp$ and the resistive part of Z/n.

Inductive part Z/n

When a bunch passes through a chamber with an inductive $Z_{||}$ a decelerating voltage is generated by this inductance which modifies the longitudinal focussing of the beam and hence the bunch length for a given emittance. A value for Z/n can be derived from these bunch lengthening measurements. This yields an average value of 7.5 Ω for Im(Z/n) in reasonable agreement with the results of the previous paragraph.

Imaginary part of $Z_\perp$

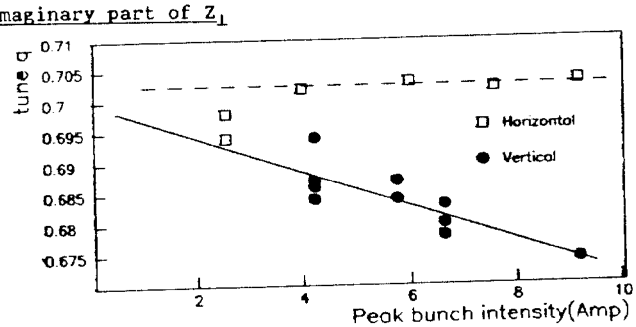

Fig. 5

The imaginary part of $Z_\perp$ can be determined by measuring the coherent tune shift of a single bunch as a function of its intensity. These measurements were done several times in the SPS. The result is shown in figure 5. The interpretation of these data is delicate since two effects produce a linear tune shift dependence with the bunch intensity, that is to say, the imaginary part of $Z_\perp$ and the direct space charge effect [3]. In fact, the two effects add in the vertical plane and subtract in the horizontal plane. Since both effects are of the same order of magnitude the resulting effect in the H plane is very close to zero.

The tune shift dependences can be read from figure 5 horizontally $\Delta Q_H/\hat{i}=0$ and vertically $\Delta Q_V/\hat{i}= 2.8\ 10^{-3}$ where $\hat{i}$ is the peak current in the bunch. This yields $Z_{\perp V}$ = 13 MΩ/m and $Z_{\perp H}$=8 MΩ/m again in reasonable agreement with the calculations.

Beam measurement of the resistive part of Z/n

From the hardware is it not possible to make definite statements about the resistive value of Z/n. Nevertheless, it is interesting to derive the resistive part of Z/n from beam measurements and compare with the imaginary part to evaluate the quality factor of the equivalent resonator. The experimental results from [4] and [5] are used in what follows. It is claimed that R(Z/n) can be derived from thresholds and growth rates of the μ-wave instability.

μ-wave measurements at 26 GeV/c

The bunch length was measured for various bunch intensities. These data are used in the following way to derive R(Z/n). The bunch is supposed to blow up until it is stable with respect to the μ-wave instability. The stability criterion involves the knowledge of dp/p, the corresponding half bunch length which together determine the bunch area, and a factor derived from the stability diagram (local application of the Keil-Schnell criterion). This determines the amplitude of the impedance vector in the stability diagram which is just stable. The average value of R(Z/n) turns out to be 40 Ω.

Conclusion

The inductive impedances derived from beam measurements in the SPS are in extremely good agreement with the values derived for the hardware using the computer code presented in this paper. The total equivalent impedance of the SPS is a broadband resonator with a quality factor of around 6. This is based on the value of the shunt impedance calculated from measurements of the μ-wave instability. The agreement with the widely used model of a Q = 1 resonator is fair.

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