NONLINEAR BEHAVIOR OF THE LONGITUDINAL MODES OF THE COASTING BEAM IN A STORAGE RING

S. A. Bogacz and K-Y Ng
Accelerator Theory Department,
Fermi National Accelerator Laboratory*
P.O. Box 500, Batavia, IL 60510

A simple nonlinear model of a coasting beam coupled to a sharp storage ring-impedance is studied in the framework of the Vlasov equation. In the case of a Gaussian beam this *non-perturbative* formalism simplifies to a pair of equations of motion which together with the dispersion relation fully describe nonlinear saturation of initially unstable coherent mode. This, in turn, provides a stabilizing mechanism (via Landau damping) for the overall distribution function. Finally, some predictions about the the energy overshoot are made.

Introduction

Various linear theories give the correct analytic description of short-time evolution of coherent instabilities, e.g. in terms of the initial growth-rate. However, this quantity fails to characterize longer time scales, i.e., when the growing instability can no longer be considered as a small fluctuation of the overall particle distribution. In order to go beyond short-time evolution studies of collective modes, one has to develop a non-linear description of the beam dynamics.

Following the arguments of Chin et al.[1], when the initial amplitude of the coherent mode is small and the instability does not develop too rapidly, one can assume that the nonlinearity modifies the particle distribution at a rate much smaller than the linear response of the system. Under this adiabaticity assumption one can formulate instantaneous dispersion relation,[2] similar to the one employed in the linear theories.

Here we apply a *non-perturbative* approach to the Vlasov equation. The resulting formalism describes the long-time behavior of driven coherent modes, their saturation due to the increasing Landau damping mechanism, and finally, how they modify the uniform part of the density distribution. Some predictions about the the energy overshoot are made on the basis of presented scheme.

Theoretical Approach

Consider a beam of particles inside a storage ring modeled by the following statistical density distribution function

$$f(\varepsilon,\theta,t) = f^{o}(\varepsilon,t) + \sum_{n\neq 0} h_n(\varepsilon,t)\, e^{i\theta n} \quad , \tag{1}$$

where θ is the azimuthal angle around the ring and ε represents the energy deviation from its synchronous value, E_o. Fourier series representation guarantees periodicity of the distribution, while the condition

$$h_{-n}(\varepsilon,t) = h_n^*(\varepsilon,t) \ , \tag{2}$$

assures that our density distribution is a real quantity. The Vlasov equation which governs $f(\varepsilon,\theta,t)$ can be written as follows

$$\frac{\partial}{\partial t} f(\varepsilon,\theta,t) + \omega \frac{\partial}{\partial \theta} f(\varepsilon,\theta,t) + \dot{\varepsilon}\, \frac{\partial}{\partial \varepsilon} f(\varepsilon,\theta,t) = 0 \quad . \tag{3}$$

Here $\omega = \omega_o + k_o\varepsilon$ and $k_o = -\eta\beta^{-2}\omega_o/E_o$ ($\eta < 0$ below the transition).

The beam environment is modeled by the wake-field of the storage ring represented in the frequency domain by the coupling impedance $Z(\omega)$. In turn, the non-uniform current coupling induces additional potential,[3] changing the energy of the beam by

$$\dot{\varepsilon} = -e\,\omega_o \sum_{n\neq 0} Z_n \phi_n(t)\, e^{i\theta n} \quad ,$$

where (4)

$$\phi_n(t) = e\,\omega_o \times \frac{1}{2\pi} \int_{-\infty}^{\infty} d\varepsilon\, h_n(\varepsilon,t)$$

and

$$Z_n = Z(n\omega_o).$$

Causality is built into $Z(\omega)$ through the Kramers-Kronig relationship ($Z_n^* = Z_{-n}$) which, together with Eq.(2), assures that $\dot{\varepsilon}$ is also a real quantity. Substituting Eqs.(1) and (4) into Eq.(3) and using orthogonality of azimuthal plane waves, one can rewrite the Vlasov equation as a set of coupled equations of motion for individual azimuthal harmonics. We shall emphasize that the equations resulting from our *non-perturbative* treatment of the Vlasov equation fully describe nonlinear behavior of the beam-storage ring system. They are given by

$$\frac{\partial}{\partial t} f^{o}(\varepsilon,t) - e\,\omega_o \sum_{n\neq 0} Z_n^*\, \phi_n^*(t)\, \frac{\partial}{\partial \varepsilon} h_n(\varepsilon,t) = 0, \tag{5}$$

and

$$\frac{\partial}{\partial t} h_n(\varepsilon,t) + in(\omega_o + k_o\varepsilon)\, h_n(\varepsilon,t) - e\,\omega_o\, Z_n\, \phi_n(t)\, \frac{\partial}{\partial \varepsilon} f^{o}(\varepsilon,t)$$
$$- e\,\omega_o \sum_{m\neq 0} Z_{n-m}\, \phi_{n-m}(t)\, \frac{\partial}{\partial \varepsilon} h_m(\varepsilon,t) = 0 \quad . \tag{6}$$

One can notice that for a high-Q impedance, $Z(\omega)$, its real part is sharply peaked around a single harmonics, $n \sim 10^4$, and is extending over several, ΔN, neighboring amplitudes. This implies that the last term in Eq.(6) would couple pairs of modes h_{n+k} and h_{-k}, where $k \leq \Delta N/2$. However Z_k is vanishingly small, therefore modes with low k will not be driven by the impedance which justifies why the last term in Eq.(6) can be neglected for our model impedance.

As we mentioned before, one can introduce instantaneous coherent frequency, $\Omega_n(t)$, describing evolution of the n-th mode within a small time interval (t, t') according to the formula

$$h_n(\varepsilon,t') = e^{-i\,\Omega_n(t)(t-t')}\, h_n(\varepsilon,t), \qquad t \approx t' . \tag{7}$$

Making a simple adiabatic approximation:

* Operated by the Universities Research Association, Inc., under a contract with the U.S. Department of Energy.

CH2387-9/87/0000-1201 $1.00 © IEEE
PAC 1987

$$\frac{\partial}{\partial\varepsilon} f^o(\varepsilon,t) = \frac{\partial}{\partial\varepsilon} f^o(\varepsilon,t'), \qquad t \approx t', . \tag{8}$$

one can rewrite Eq.(6) as follows

$$h_n(t) = (e\,\omega_o)^2 \frac{\partial}{\partial\varepsilon} f^o(\varepsilon,t) \frac{Z_n}{n\omega - \Omega_n} \frac{1}{2\pi i} \int_{-\infty}^{\infty} d\varepsilon'\, h_n(\varepsilon',t). \tag{9}$$

As was pointed out by Landau,[2] an appropriate integration of Eq.(9) over ε leads to the following dispersion relationship

$$1 = (e\omega_o)^2 \frac{N Z_n}{2\pi n k_o} \frac{1}{2\pi i} \int_C d\varepsilon \frac{\frac{\partial}{\partial\varepsilon} g^o(\varepsilon,t)}{\varepsilon - \xi_n} . \tag{10}$$

Here, $\xi_n = (\Omega_n/n - \omega_o)/k_o$ fixes the pole in the complex ε-plane, while C is the Landau contour.[2] We also replaced $f^o(\varepsilon,t)$ with a normalized distribution function;

$$g^o(\varepsilon,t) \equiv \frac{2\pi}{N} f^o(\varepsilon,t), \tag{11}$$

where N is the number of particles in the ring.

From here on in, we will confine our discussion to a single mode, h_n, driven by the model impedance, i.e. index n will be suppressed throughout the rest of the paper. The summation over all modes in Eq.(5) reduces to two terms only (n and $-n$) which combined with the symmetry condition, Eq.(3), yields the following formula

$$\frac{\partial}{\partial t} g^o(\varepsilon,t) - \frac{2\pi}{N} e\,\omega_o\, 2\,\mathrm{Re}\{ Z^* \phi^*(t) \frac{\partial}{\partial\varepsilon} h(\varepsilon,t) \} = 0 \ . \tag{12}$$

We can easily generalize the above result to the case of impedance extending over several, ΔN, azimuthal harmonics. Simply, replacing summation over n in Eq.(5) by integration, carrying it out and retaining only the leading, $\Delta N/N$, order one obtains Eq.(12) with Z replaced by $Z\Delta N$. The last expression is obviously proportional to the area under the peak which assures the correct scaling of our result.

Now, we make use of the fact that the distribution function, $g^o(\varepsilon,t)$, is uniquely defined by an infinite set of its moments with respect to ε. Introducing the following notation

$$G_k(t) \equiv \int_{-\infty}^{\infty} d\varepsilon\, g^o(\varepsilon,t)\, \varepsilon^k$$

and (13)

$$H_k(t) \equiv \int_{-\infty}^{\infty} d\varepsilon\, h(\varepsilon,t)\, \varepsilon^k ,$$

one can rewrite Eq.(12) as a simple set of equations of motion for ε-moments, $G_k(t)$,

$$\frac{\partial}{\partial t} G_o(t) = 0 \ , \quad G_o = 1 \quad \text{(normalization)},$$

$$\frac{\partial}{\partial t} G_m(t) - \frac{m}{N} (e\omega_o)^2\, 2\mathrm{Re}\{ Z^* H_o^*(t) H_{m-1}(t) \} = 0, \ m \ge 1. \tag{14}$$

We observe that integrating Eq.(9) along C, after some algebra and integration by parts, one obtains the desired recursion formula

$$H_{m-1} = \{ \xi^{m-1} - (e\,\omega_o)^2 \frac{NZ}{2\pi n k_o} \frac{1}{2\pi i} \sum_{k=1}^{m-2} k\,\xi^{m-k-2} G_{k-1} \} H_o. \tag{15}$$

Final substitution of Eq.(15) into Eqs.(14), allows us to rewrite them as follows

$$\frac{\partial}{\partial t} G_1(t) + 2\pi e\,\omega_o I_o\, |H_o/N|^2\, 2\,\mathrm{Re}\{ Z \} = 0 ,$$

$$\frac{\partial}{\partial t} G_2(t) + 4\pi e\,\omega_o I_o\, |H_o/N|^2\, 2\,\mathrm{Re}\{ \xi Z^* \} = 0 , \tag{16}$$

$$\vdots$$

$$\frac{\partial}{\partial t} G_m(t) + 2m\pi e\,\omega_o I_o\, |H_o/N|^2\, 2\,\mathrm{Re}\{ \xi^{m-1} Z^* - \frac{e\,\omega_o I_o |Z|^2}{2\pi n k_o} \times [\xi^{m-3} + 2\xi^{m-4} G_1(t) + 3\xi^{m-5} G_2(t) + \dots] \} = 0.$$

Here, $I_o = Ne\omega_o/2\pi$ represents the current in the storage ring. The time evolution of the coherent mode amplitude, $A(t) \equiv |H_o(t)/N|^2$, is governed by $\Omega(t)$, through the following equation

$$\frac{\partial}{\partial t} A(t) - 2\,\mathrm{Im}\{ \Omega(t) \} A(t) = 0 , \tag{17}$$

which is the immediate consequence of Eq.(7).

One can summarize our scheme by realizing that an infinite system of coupled equations, Eqs.(16), together with Eq.(17) and the dispersion relation, Eq.(10), form a closed set of equations which will be further simplified in the case of a Gaussian beam in the next section.

Gaussian Beam Dynamics

For the purpose of our model calculation, we will start with a Gaussian beam coasting in a storage ring. We assume that the distribution maintains its initial Gaussian shape with the time dependent parameters, $M = M(t)$ and $S = S(t)$

$$g^o(u,t) = (\alpha/\pi)^{1/2} \exp\{ -\alpha(u - M)^2 \} , \tag{18}$$

where $\alpha = 1/(2S^2)$ and $u = \varepsilon/E_o$

The above assumption is justified by studying the skewness of the distribution, defined as, $Q \equiv G_3/E_o^3 - M(M^2 + 3S^2)$. The vanishing of this characteristic guarantees that the beam, indeed, retains its initial shape. The following obvious identification

$$G_1/E_o = M$$

and (19)

$$G_2/E_o^2 = S^2 + M^2 ,$$

allows us to simplify the second and third equations of motion, Eqs.(16), by introducing the convenient dimensionless quantities

$$\mathcal{Z} \equiv e I_o Z/E_o$$

and (20)

$$x \equiv \xi/E_o .$$

These equations can be rewritten as

$$\frac{\partial}{\partial t} M(t) + 2\pi\,\omega_o A(t)\, 2\,\mathrm{Re}\{ \mathcal{Z} \} = 0$$

and (21)

$$\frac{\partial}{\partial t} S^2(t) + 4\pi\,\omega_o A(t)\, 2\,\mathrm{Re}\{ (x - M)\mathcal{Z}^* \} = 0 .$$

Now our problem is reduced to a self-consistent solution of Eq.(17) and Eqs.(21) with the coherent frequency, x, defined implicitly by the integral dispersion formula, Eq.(10). This system of nonlinear integral equations is no longer tractable analytically, nevertheless, its time evolution can be easily iterated numerically assuming the following initial condition of our system; $M = 0$, $S = S_0$.

Energy Overshoot

As a simple application of our formalism, one can study nonlinear saturation effects contributing to the overshoot of the energy spread. Assumed beam-storage ring parameters are collected in a table below:

I_o	E_o	Z/n	n
10^{-3} Amp.	100 GeV	10^8 Ohm.	10^4

The above values fix the instability threshold at $S_{th} = 3.33\times10^{-4}$. In order to start with an initially growing non-uniform mode, one has to select S_0 below S_{th}. The intrinsic amplitude of the coherent mode, A_0, is assigned an arbitrary small value of 10^{-18} which sets the level of Schottky noise in the system. The result for $S_0 = 10^{-4}$ is illustrated in Figs.1 and 2. One can see that the coherent mode of an arbitrarily small amplitude, A_0, is growing initially very fast, according to Eq.(17). Its growth, in turn, causes increase of the energy spread, S, and a negative shift of the distribution mean value, M, (energy losses due to the resistive storage ring impedance). This affects the coherent frequency through the dispersion relation; the new values of S and M correspond to stronger Landau damping which results in a succesive decrease of the growth-rate, $Im(\Omega)$. Finally, the coherent frequency crosses into the stable region, $Im(\Omega) < 0$, which triggers rapid decay of the coherent mode. This eventually stabilizes all characteristics S, M, and G since the amplitude, A, goes exponentially back to zero. After saturation, both Eqs.(21) approach asymptotically their stationary solutions S_∞ and M_∞. We notice in passing that the choice of intrinsic small amplitude, A_0, has very little influence on the curves presented in Figs.1 and 2 (as long as $A_0 << A_{max}$). Going to smaller values of A_0 does not change the shape of presented beam characteristics; it only shifts them in time (it takes longer for the instability to develop).

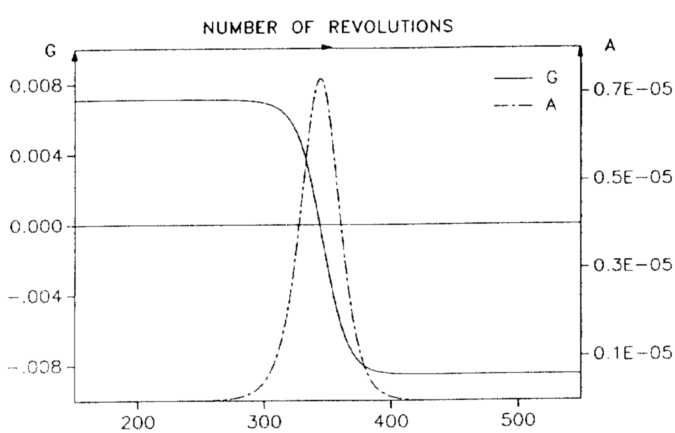

Fig.1 Nonlinear Landau damping mechanism - single mode coupling.

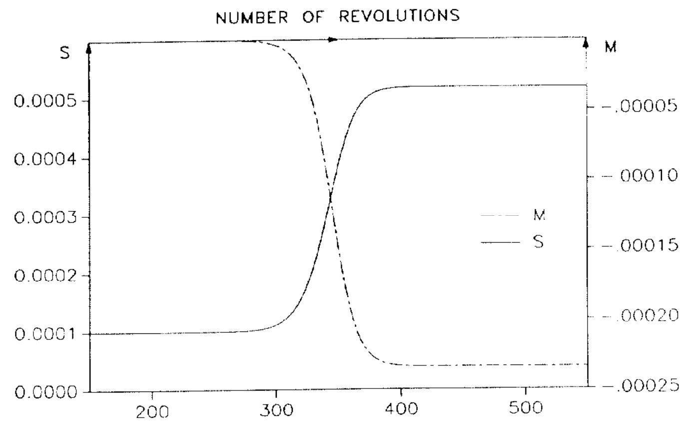

Fig.2 Saturation of the longitudinal momentum spread and the energy shift caused by nonlinear Landau damping.

The correlation between S_0 and S_∞ has been studied before; first by Dory by computer simulations and later by Chin et al.[1] by approximated analytic treatment of the Vlasov equation. Applying our formalism, the stationary values, S_∞, were calculated numerically for several values of S_0. The resulting energy overshoot law is compared with the ones previously formulated by Dory and Chin (Fig.3). Using the least-square-fit criterion we realize that by replacing the "square" exponent in Chin's formula $(S_\infty^2 + S_0^2 = 2S_{th}^2)$ by the exponent $a = 1.36$, one achieves a good fit to our result.

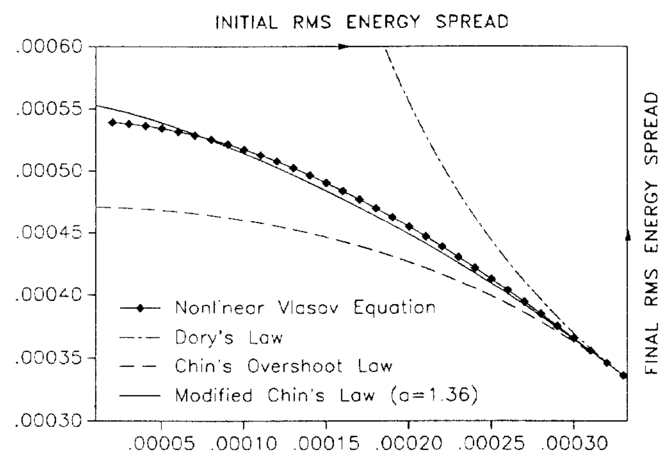

Fig.3 Nonlinear Vlasov equation approach to the energy overshoot phenomenon. Comparison with the existing results.

References

[1] Y. Chin and K. Yokoya, Phys. Rev. D, **28**, 2141, (1983)

[2] C.Lashmore-Davies, Plasma Physics and Instabilities, Ch.3 and 5, CERN 81-13, Geneva 1981

[3] S. Krinsky and J.M. Wang, Particle Accelerators, **17**, 109, (1985)