QUADRUPOLE FREE ELECTRON LASER

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Abstract

The nonlinear motion of electrons in a continuously rotating quadrupole free-electron laser is analyzed. The equations of motion are described by a three-dimensional Hamiltonian system. By applying a set of canonical transformations the three degree-of-freedom Hamiltonian is reduced to a two degree-of-freedom Hamiltonian. The effect of the betatron-synchrotron resonance in the quadrupole FEL is then analyzed.

Considerable interest has been expressed recently in the possible utilization of magnetic quadrupoles in free electron lasers for focusing of the electron beam[1-4] and for wigglers.[5-9]

Here, we analyze the nonlinear motion of electrons in a free electron laser under the combined influence of a continuously rotating quadrupole magnetic wiggler and a circularly polarized electromagnetic wave. The linear theory of this configuration has been presented previously.[6,7]

The nonlinear equations of motion of a relativistic electron in the quadrupole FEL can be deduced from the Hamiltonian

$$H = \frac{1}{2}\left(P_x^2 + P_y^2 + \frac{P_\chi^2}{m_\ell}\right) + \frac{1}{2}\alpha^2(x^2+y^2) - a(x\cos\chi + y\sin\chi) \qquad (1)$$

with

$$m_\ell = [(1+\beta_0^2)\gamma_0^2]^{-1} .$$

All quantities in (1) are dimensionless, x and y are normalized to k_q, the wiggler wavenumber. The beat wave amplitude a is given by

$$a = \frac{|e|^2 E_0 B_0}{m^2 k_q^2 c^4}\left(\frac{1-\beta_0}{\beta_0\gamma_0^2}\right) ,$$

and the wiggler strength is characterized by

$$\alpha = \frac{|e| B_0}{mc^2 k_q \gamma_0} ,$$

where E_0 is the amplitude of the electromagnetic wave; B_0 is the value of the magnetic field at the distance of one wiggler wavelength from the z-axis; β_0 and γ_0 are the resonant velocity and energy, respectively; e and m are electron charge and mass, respectively; and c is the speed of light. The quantity χ is the phase of the electron in the beat wave $\chi = \tilde{k}(z - \beta_0 t) + \phi$, where z is the axial position, $\tilde{k}$ is the normalized beat wave number $(\tilde{k} = (k_r + k_q)/k_q)$, and ϕ is the relative phase between the circularly polarized electromagnetic wave and the magnetic wiggler, time is normalized to the wiggler "frequency," $(k_q c)^{-1}$. Initially, $\dot{\chi}_0 = \tilde{k}(\beta_i - \beta_0)$, where β_i is the injection velocity in the z-direction. Using the resonance condition we can express

$$\dot{\chi}_0 = [(k_r + k_q)\beta_i - k_r]/k_q$$

and denote $\dot{\chi}_0 = \Delta\omega$ as the detuning parameter.

From the Hamiltonian it follows that if the radiation field is zero, the particles perform transverse betatron oscillations with the frequency α. By switching on the electric field, this transverse motion couples to longitudinal oscillations due to the ponderomotive force, so that, in general, the particle orbits are three-dimensional. For a long enough wiggler and for a certain range of the parameters and the initial conditions, this coupled motion may lead to a detrapping of some particles from the ponderomotive potential. Here we study this effect.

The number of free parameters in (1) can be reduced by redefining the time $t = t_R\tau$ and $(x,y) = R(\hat{x},\hat{y})$, where $t_R = (m_\ell 2\alpha/a^2)^{1/3}$ and $R = (am_\ell/4\alpha^2)^{1/3}$. The Hamiltonian now has the form

$$H = \frac{1}{2}\left(\hat{P}_x^2 + \hat{P}_y^2 + \frac{P_\chi^2}{2\kappa}\right) + \frac{1}{2}\kappa^2(\hat{x}^2+\hat{y}^2) - 2\kappa(\hat{x}\cos\chi + \hat{y}\sin\chi) , \qquad (2)$$

where $\kappa = t_R\alpha$. In terms of the polar coordinates, the Hamiltonian (2) can be written

$$H = \frac{1}{2}\left(P_r^2 + \frac{P_\theta^2}{r^2} + \frac{P_\chi^2}{2\kappa}\right) + \frac{1}{2}\kappa^2 r^2 - 2\kappa r\cos(\chi-\theta) . \qquad (3)$$

We can achieve a reduction in the degrees-of-freedom in the Hamiltonian by performing the following contact transformation, using a generating function

$$F = (\chi-\theta)P_\psi + \theta P_\phi , \qquad (4)$$

where P_ψ and P_ϕ are new canonical momenta. Making use of the standard transformation formulae relating the old to the new coordinates, we write $\phi = \theta$, $\psi = \chi - \theta$, $P_\psi = P_\chi$, and $P_\phi = P_\theta + P_\chi$. In the new coordinates, the Hamiltonian has the following form

$$H = \frac{1}{2}\left(P_r^2 + \frac{P_\psi^2}{2\kappa} + \frac{(P_\phi - P_\psi)^2}{r^2}\right) + \frac{1}{2}\kappa^2 r^2 - 2\kappa r\cos\psi \ . \tag{5}$$

It is to be noted that since the Hamiltonian (5) is independent of ϕ, the conjugate momenta P_ϕ is a constant of the motion. The reduction of the Hamiltonian (3) from three degrees-of-freedom to two degrees-of-freedom is a manifestation of the helical symmetry of the electron motion in the continuously rotating quadrupole wiggler. We call P_ϕ the helical momentum.

We can achieve a deeper insight into the problem by expressing the Hamiltonian in terms of action-angle variables. The details of the calculation will be described in the later publication. Here we present the final results; the new Hamiltonian has a form

$$H = \kappa J + \kappa(P_\phi - P_\psi) + \frac{P_\psi^2}{4\kappa}$$

$$- 2\kappa\left[\left(\frac{A+B}{2}\right)\cos\tilde{\psi} + \left(\frac{A-B}{2}\right)\cos(\tilde{\psi} + 2\rho)\right] , \tag{6}$$

where J and ρ are action, angle coordinates $J = \frac{1}{2}\kappa(A+B)$, and $\tilde{\psi} = \psi - \rho + \mathrm{arctg}\,\frac{B}{A}\,\mathrm{tg}\rho$, with

$$A^2 = |\xi_+|^2 + |\xi_-|^2 - |\xi_-||\xi_+|$$

and

$$B^2 = |\xi_+|^2 + |\xi_-|^2 + |\xi_+||\xi_-| \ ,$$

where

$$|\xi_+|^2 = \{P_r^2 + [r\kappa + (P_\phi - P_\psi)/r]^2\}/4\kappa^2$$

and

$$|\xi_-|^2 = \{P_r^2 + [r\kappa - (P_\phi - P_\psi)/r]^2\}/4\kappa^2 \ . \tag{7}$$

The complex functions ξ_+ and ξ_- are related to the transverse motion of the particles by the formula

$$\xi(\tau) = \xi_+(\tau)e^{i\kappa\tau} + \xi_-(\tau)e^{-i\kappa\tau} \tag{8}$$

where

$$\xi = re^{i\theta} \ .$$

Using (6), we estimate the location of the two primary resonances $\dot{\tilde{\psi}} \simeq 0$ and $\dot{\tilde{\psi}} + 2\dot{\rho} \simeq 0$, namely

$$P_{\psi,\pm} \simeq \pm 2\kappa^2 \ . \tag{9}$$

This result was evidenced in the linear theory[6] where it was observed that the gain peaked when the mismatch frequency $\Delta\omega$ coincided with the betatron frequency $\Delta\omega \pm \alpha \simeq 0$.

Assuming that the resonances are well separated and κ is big, then in the neighborhood of $\dot{\tilde{\psi}} \simeq 0$, the Hamiltonian (6) can be approximated

$$H \simeq \kappa J + \kappa(P_\phi - P_\psi) + \frac{P_\psi^2}{4\kappa} - \kappa(A + B)\cos\tilde{\psi} \ . \tag{10}$$

Similarly, in the neighborhood of $\dot{\tilde{\psi}} + 2\dot{\rho} \simeq 0$, we obtain

$$H \simeq \kappa J + \kappa(P_\phi - P_\psi) + \frac{P_\psi^2}{4\kappa} - \kappa(A-B)\cos(\tilde{\psi} + 2\rho) \ . \tag{11}$$

The widths of the resonances are

$$\Delta P_{\psi\pm} = 4\kappa\left(\frac{B \pm A}{2}\right)^{1/2} \ . \tag{12}$$

Now we can apply Chirikov resonance overlap criterion[10] to estimate the onset of global stochasticity. According to this criterion, two resonances will overlap, provided that the sum of their respective halfwidths is larger than their relative spacing. In our case it follows from (9) and (12) that

$$S = [(A+B)/2]^{1/2} + [(B-A)/2]^{1/2} \gtrsim 2\kappa \ . \tag{13}$$

More precise study of the overlap requires numerical solution of Hamilton's equations using surface of section techniques. We choose Hamiltonian (5) to generate the equations of motion. We assume electrons are injected with $P_r = 0$, $P_\psi = P_\phi$, $r = 2.8$, and ψ uniformly distributed between 0 and 2π initially. We record the points of intersection of a given trajectory with the plane $P_r = 0$ (only the points with $\dot{P}_r > 0$ are recorded). The surface of section for the case of $\kappa = 10$ is shown in Fig. (1a). It is seen that the orbits in this case lie on smooth curves implying the existence of an additional conserved quantity. In fact, in the case of large κ the equations of motion may be further reduced and solved by quadrature.[9] This third constant of motion (in addition to P_ϕ and H) turns out to be approximately $|\xi_+|^2$ for $P_\psi \simeq -2\kappa^2$ and $|\xi_-|^2$ for $P_\psi \simeq +2\kappa^2$. The behavior pictured here is qualitatively the same as that predicted by the pendulum equation. As κ is decreased we note that the motion for some initial conditions becomes irregular. Figures 1b and 1c show the cases of $\kappa = 4$ and $\kappa = 2$, respectively. For $\kappa = 4$, there is a region where all the orbits are bounded. A six-island chain separates this region from the stochastic region, and then particles with the circulating orbits appear. For $\kappa = 2$ all the orbits are stochastic.

In principle, particles can remain trapped even though their trajectories are stochastic. This occurs because conservation of the Hamiltonian restricts the values of ψ which the particles can have. Thus, for example, if $\psi = \pm \pi$ is inaccessible, the particle will remain trapped. There are other particles which follow a stochastic orbit and become untrapped, but the changes in ψ remain modest. The above study is only relevant if a tapered quadrupole wiggler is under consideration. If a variable parameter wiggler is introduced the helical momentum P_ϕ is no longer a constant and the 3-D Hamiltonian can not be reduced to the 2-D Hamiltonian. However, as we showed before,[9] if the conditions for the $P_\psi \approx 2\kappa^2$ or $P_\psi \approx -2\kappa^2$ resonance are satisfied, the particles if initially trapped will stay trapped.

Thus we found that for large values of κ the electrons initially trapped in the beat wave potential well will remain trapped. This is consistent with the estimate. Approximating the left-hand side of (13) by $\sqrt{r}$ and using the formulas for κ, t_R, and R and the definitions for a, m_ℓ, and α, Eq. (13) becomes

$$(1 + \beta_0^2)/(1 + \beta_0)\, \frac{1}{4}\, (\kappa_q r_b) \gtrsim (B_0/E_0)\ , \qquad (14)$$

where r_b is the electron beam radius.

For most practical cases (14) is very difficult to satisfy. Therefore, within the bounds of the physical model the betatron-synchrotron resonance in the quadrupole FEL will not introduce significant threat for the operation.

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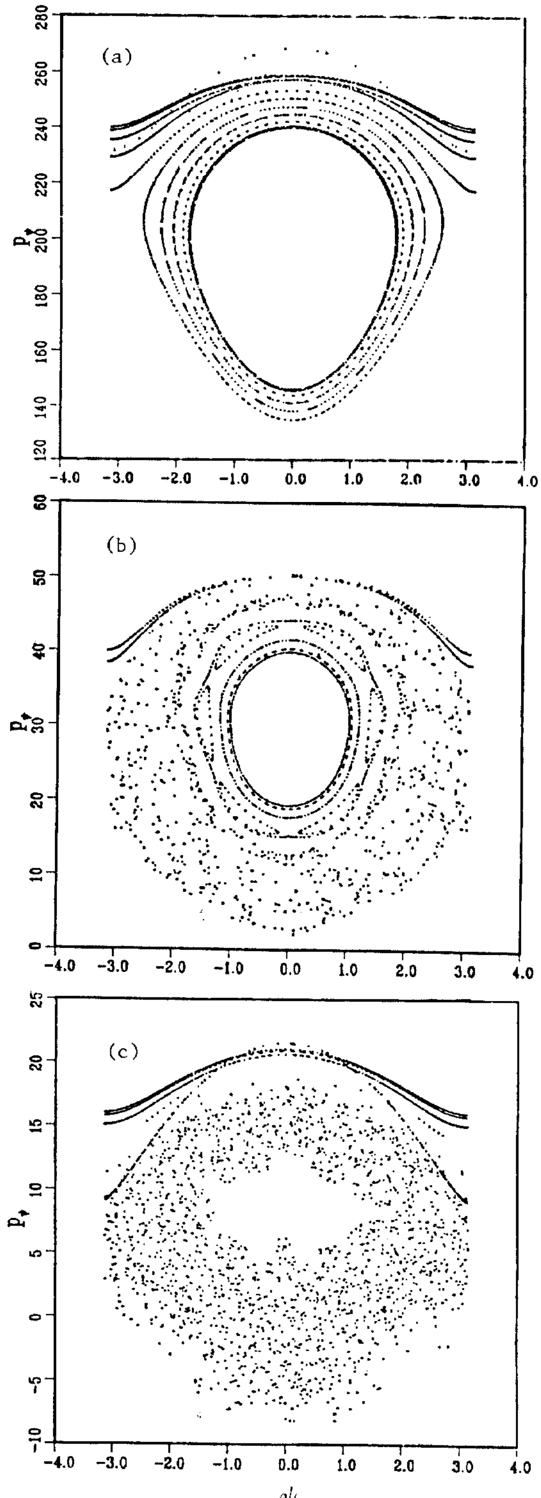

FIG. 1