A semi-analytical approach to six-dimensional path-dependent transport matrices with application to high-brightness charged-particle beam transport

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Abstract: Efficient and accurate estimate of high-brightness electron beam dynamics is an important step to the overall performance evaluation in modern particle accelerators. Utilizing the moment description to study multi-particle beam dynamics, it is necessary to develop a path-dependent transport matrix, together with application of the drift-kick algorithm. In this work we construct semi-analytical models for two typical beam transport elements, **solenoid** with fringe fields and transverse deflecting cavity (**TDS**). To construct the semi-analytical models for these elements, we begin by formulating the simplified single-particle equations of motion, and apply numerical techniques to solve the corresponding six-by-six transport matrix as a function of the path coordinate. The developed semi-analytical models are demonstrated with practical examples, where the numerical results are discussed, compared with and validated by particle tracking simulations. These path-dependent transport matrix models can be incorporated to the analysis based on beam matrix method for the application to high-brightness charged-particle beam transport.

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Beam moment description for multi-particle dynamics

- Pure-optics linear transport

$$\frac{d\mathbf{X}}{ds} = \mathbf{J}\frac{\partial \mathcal{H}}{\partial \mathbf{X}} = \mathbf{A}\mathbf{X}$$

$$\frac{d\mathbf{M}}{ds} = \mathbf{A}\mathbf{M}$$

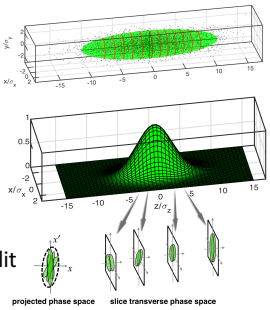
Vlasov equation

$$\frac{df}{ds} = \frac{\partial f}{\partial s} + \left(\frac{\partial f}{\partial \mathbf{X}}\right)^T \frac{d\mathbf{X}}{ds} = 0$$

centroid & envelope equation

$$\langle \mathbf{X}(s_f)\rangle = \mathbf{M}(s_i \to s_f)\,\langle \mathbf{X}(s_i)\rangle$$

$$\boldsymbol{\Sigma}(s_f) = \mathbf{M}(s_i \to s_f)\boldsymbol{\Sigma}(s_i)\mathbf{M}^T(s_i \to s_f)$$



- Space charge model, bulk and slice beam matrix description and propagation: C.-Y. Tsai *et al.*, NIMA **937** 1-20 (2019)
  - ✓ available transport elements: RF cavity, sector/rectangular dipole, quadrupole, hard-edge solenoid, beam slit
- *s*-dependent transport matrix is **required** for high-brightness beam dynamics study

Semi-analytical models for solenoid with fringe fields

Assume the longitudinal *B* field is collinear with z

$$\mathbf{B} = \left(-\frac{1}{2}B_s'x, -\frac{1}{2}B_s'y, B_s\right)$$

within transverse 4D phase space coordinate (x,x',y,y')

$$x'' = \mathcal{S}(s)y' + \frac{1}{2}\mathcal{S}'(s)y, \qquad \mathcal{S}(s) = eB_s(s)/\gamma m\beta c = B_s(s)/[B\rho]$$

$$y'' = -\mathcal{S}(s)x' - \frac{1}{2}\mathcal{S}'(s)x,$$

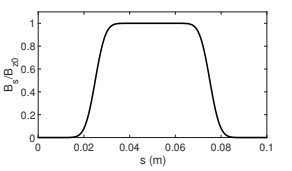
Transform to Larmor frame,

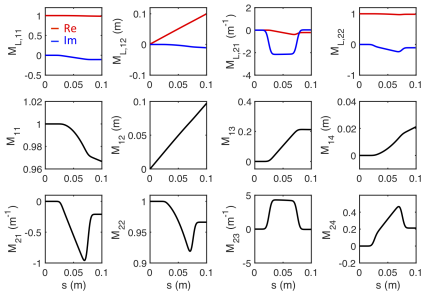
$$\frac{d}{ds}\begin{pmatrix}\eta_L\\ \eta_L'\end{pmatrix} = \begin{pmatrix}0 & 1\\ -\frac{i}{2}\mathcal{S}'(s) & -i\mathcal{S}(s)\end{pmatrix}\begin{pmatrix}\eta_L\\ \eta_L'\end{pmatrix} = \mathbf{A}\begin{pmatrix}\eta_L\\ \eta_L'\end{pmatrix}$$

Given **A**, we can solve for **M** at each *s* (in Larmor frame), then transform back to Lab frame by

$$\mathbf{R}\,(\text{Lab}\to\text{Larmor}) = \begin{pmatrix}\cos\Delta\theta_L & 0 & \sin\Delta\theta_L & 0\\ -\Delta\theta_L'\sin\Delta\theta_L & \cos\Delta\theta_L & \Delta\theta_L'\cos\Delta\theta_L & \sin\Delta\theta_L\\ -\sin\Delta\theta_L & 0 & \cos\Delta\theta_L & 0\\ -\Delta\theta_L'\cos\Delta\theta_L & -\sin\Delta\theta_L & -\Delta\theta_L'\sin\Delta\theta_L & \cos\Delta\theta_L\end{pmatrix}$$

Note: the longitudinal components are treated as a drift

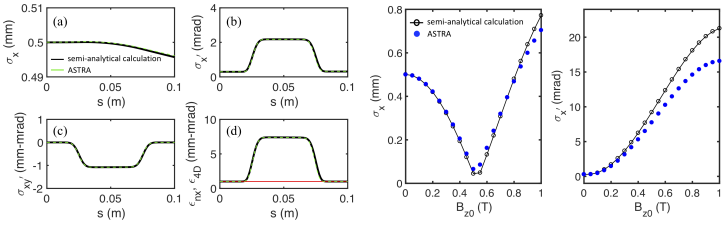




$$B_s(s) = B_{z0}\frac{\text{erf}\left(\frac{s-L_{sol}/4}{\kappa}\right) - \text{erf}\left(\frac{s-3L_{sol}/4}{\kappa}\right)}{2\text{erf}\left(\frac{L_{sol}}{2\kappa}\right)}$$

$\kappa = 0.01$, $B_{z0} = 0.05$ T

3 MeV, $\sigma_x = 0.5$ mm, $\varepsilon_{nx} = 1\ \mu$m



Semi-analytical models for transverse deflecting cavity

Assume TM_{110} mode

$$E_z = -\frac{2B_0\omega}{\kappa}J_1(\kappa r)\cos\theta\sin(\omega t+\phi_0),$$
$$B_r = \frac{2B_0}{\kappa r}J_1(\kappa r)\sin\theta\cos(\omega t+\phi_0),$$
$$B_\theta = 2B_0J'_1(\kappa r)\cos\theta\cos(\omega t+\phi_0),$$

$$\Rightarrow \quad B_y = B_0\cos(\omega t+\phi_0), \qquad E_z = -B_0\omega x\sin(\omega t+\phi_0)$$

From Lorentz equation, we have solve 6-D phase space coordinates for representative particles

$$\frac{dp_x}{dt} = ev_zB_0\cos(\omega t+\phi_0), \quad \frac{dp_y}{dt} = 0,$$
$$\frac{dp_z}{dt} = eB_0\omega x\sin(\omega t+\phi_0) + ev_xB_0\cos(\omega t+\phi_0),$$
$$\frac{dx}{dt} = \frac{p_x}{\gamma m}, \frac{dy}{dt} = \frac{p_y}{\gamma m}, \frac{dz}{dt} = \frac{p_z}{\gamma m}.$$

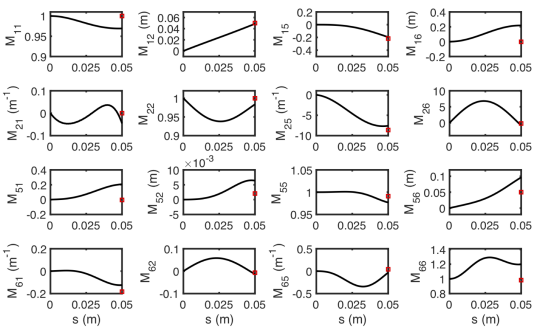
Extract the 6x6 transport matrix elements via $\mathbf{Z}^i_{16\times16}\mathbf{R}^{i\to f}_{16\times1}(s) = \mathbf{Z}^f_{16\times1}(s)$

$$\begin{bmatrix}x\\ x'\\ z\\ z'\end{bmatrix}_s = \begin{pmatrix}M_{11}^{s_i\to s} & M_{12}^{s_i\to s} & M_{15}^{s_i\to s} & M_{16}^{s_i\to s}\\ M_{21}^{s_i\to s} & M_{22}^{s_i\to s} & M_{25}^{s_i\to s} & M_{26}^{s_i\to s}\\ M_{51}^{s_i\to s} & M_{52}^{s_i\to s} & M_{55}^{s_i\to s} & M_{56}^{s_i\to s}\\ M_{61}^{s_i\to s} & M_{62}^{s_i\to s} & M_{65}^{s_i\to s} & M_{66}^{s_i\to s}\end{pmatrix}\begin{bmatrix}x\\ x'\\ z\\ z'\end{bmatrix}_{s_i}$$

$$\mathbf{Z}^i_{16\times16} = \begin{pmatrix}\mathcal{Z}^{(1-4)}_{4\times4} & 0 & 0 & 0\\ 0 & \mathcal{Z}^{(5-8)}_{4\times4} & 0 & 0\\ 0 & 0 & \mathcal{Z}^{(9-12)}_{4\times4} & 0\\ 0 & 0 & 0 & \mathcal{Z}^{(13-16)}_{4\times4}\end{pmatrix}$$

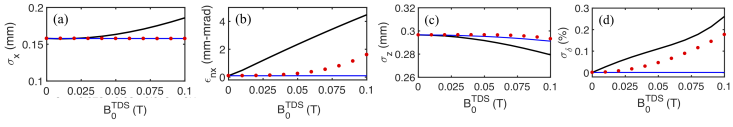
$$\mathbf{X}^{(i)}_{4D} = \left(x^{(i)}, x'^{(i)}, z^{(i)}, z'^{(i)}\right)$$

$$\mathcal{Z}^{(1-4)}_{4\times4} = \left[\mathbf{X}^{(1)}_{4D}\ \mathbf{X}^{(2)}_{4D}\ \mathbf{X}^{(3)}_{4D}\ \mathbf{X}^{(4)}_{4D}\right]^T$$



Analytical results by van Rens et al., Ultramicroscopy **184** 77-89 (2018)

Tracking results by ELEGANT



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