ISBN: 978-3-95450-223-3 ISSN: 2673-7000 doi:10.18429/JACoW-NAPAC2019-WEPLS03

ANALYTICAL EXPRESSION FOR A N-TURN TRAJECTORY IN THE PRESENCE OF QUADRUPOLE MAGNETIC ERRORS

Y. Rodríguez García,[1]* J. F. Cardona †, Universidad Nacional de Colombia, Bogotá, Colombia
[1] also at Universidad Antonio Nariño, Bogotá, Colombia

Abstract

The action and phase jump method is a technique, based on the use of turn-by-turn experimental data in a circular accelerator, to find and measure local sources of magnetic errors through abrupt changes in the values of action and phase. At this moment, this method uses at least one pair of adjacent BPMs (Beam Position Monitors) to estimate the action and phase at one particular position in the accelerator. In this work, we propose a theoretical expression to describe the trajectory of a charged particle for an arbitrary number of turns when a magnetic error is present in the accelerator. This expression might help to estimate action and phase at one particular position of the accelerator using only one BPM in contrast to the current method that needs at least two BPMs

INTRODUCTION

The Action and Phase Jump Analysis Technique, known as APJ method, is one of the available methods to estimate local magnetic field errors. This method uses as a theoretical argument that the action and phase in betatron oscillations must be preserved in the absence of a magnetic error. This method requires at least two adjacent BPMs to estimate the action and phase variables as described in [1, 2].

In this paper, an analytical expression to describe a N-turn trajectory is proposed considering the presence of a magnetic error, which might provide an alternative way to compute action and phase using only one BPM. First, the betatron oscillations in an arbitrary BPM are described through a first-order approximation for betatron oscillations in that place where a magnetic error is present. Then, an improved expression is obtained using perturbation theory. Finally, the resulting expression is compared with simulated turn by turn trajectories.

TURN BY TURN TRAJECTORY

Reference [1] shows that one-turn trajectory after the particle has passed through a magnetic error can be described by

$$\begin{aligned} z(s) &= \sqrt{2J_{z0}\beta_z(s)}\sin\left[\psi_z(s)-\delta_{z0}\right] \\ &\quad + \theta_z\sqrt{\beta_z(s)\beta_z(s_\theta)}\sin\left[\psi_z(s)-\psi_z(s_\theta)\right] \\ &= \sqrt{2J_{z1}\beta_z(s)}\sin\left[\psi_z(s)-\delta_{z1}\right] \end{aligned} \tag{1}$$

where z denotes either the x or y axis, β_z represent the nominal beta functions, J_{z_0} (δ_{z_0}) and J_{z_1} (δ_{z_1}) are the actions (phases) before and after the error, while ψ_z represents the betatron function. Both J and δ remain constant except in the error position, represented by s_θ, where they suffer an abrupt jump. The strength of the magnetic error θ_z can be of any order: a dipole, a quadrupole, etc., and it can be estimated by (15) from [1].

For the case in which there is more than one error, say m, the left side of (1) takes the form given by

$$\begin{aligned} z(s) &= \sqrt{2J_{z0}\beta_z(s)}\sin\left[\psi_z(s)-\delta_{z0}\right] \\ &\quad + \sum_{i=1}^{m}\theta_{zi}\sqrt{\beta_z(s)\beta_z(s_i)}\sin\left[\psi_z(s)-\psi_z(s_i)\right], \end{aligned} \tag{2}$$

where subscript i denotes the *i-th* magnetic error located at $s = s_i$. It is easy to show that the value of z at any fixed BPM placed in the longitudinal position s for a given number of turns n can be calculated as follows

$$\begin{aligned} z(s,n) &= \sqrt{2J_{z0}\beta_z(s)}\sin\left[\psi_z(s)-\delta_{z0}+2\pi Q_z(n-1)\right] \\ &+ \sum_{j=1}^{n}\sum_{i=1}^{m}\theta_{zi,j}\sqrt{\beta_z(s)\beta_z(s_i)} \\ &\quad \cdot\sin\left[\psi_z(s)-\psi_z(s_i)+2\pi Q_z(n-j)\right], \end{aligned} \tag{3}$$

with Q_z being the nominal tune value and j the subscript indicating the number of the turn. In the following and without loss of generality, the orbit given by Eq. (3) will be referred to the case $z = x$.

APPROXIMATION AT FIRST ORDER

According to equation (19) from [1], θ_{xi} can be written from the multipolar expansion of the magnetic field as follows

$$\begin{aligned} \theta_{xi} &= B_{0i} - B_{1i}x(s_i) + A_{1i}y(s_i) + 2A_{2i}x(s_i)y(s_i) \\ &\quad + B_{2i}\left[-x^2(s_i)+y^2(s_i)\right] + \cdots \end{aligned} \tag{4}$$

where B_{ki} and A_{ki} correspond to the integrated skew and normal quadrupole components of the *i-th* magnetic error, and $x(si)$, $y(si)$ are the transverse coordinates of the orbit at the error location. If only normal quadrupole errors are considered, the Eq. (4) is reduced to

$$\theta_{xi} = -B_{1i}x(s_i). \tag{5}$$

In practice, Eq. (3) could theoretically reproduce the trajectory for n turns if $x(s_i)$ values in Eq. (5) were available,

* yrodriguezga@unal.edu.co
† jfcardona@unal.edu.co

ISBN: 978-3-95450-223-3 ISSN: 2673-7000 doi:10.18429/JACoW-NAPAC2019-WEPLS03

but they should also be determined by Eq. (3). This leads to a process of recursion that is difficult to generalize. The proposed solution consists of an approximation for $x_j(s_i)$ obtained from (1) and (3), which is given by

$$\begin{aligned} x_j(s_i) &= \sqrt{2J_{x_0}\beta_x(s_i)}\sin\left[\psi_x(s)-\delta_{x_0}+2\pi Q_x(j-1)\right] \\ &+ \theta_{xi,j}\sqrt{\beta_x(s)\beta_x(s_i)}\sin\left[\psi_x(s)-\psi_x(s_i)\right] \\ &\approx x_j(s_i) = \sqrt{2J_{x_0}\beta_x(s_i)}\sin\left[\psi_x(s)-\delta_{x_0}+2\pi Q_x(j-1)\right], \end{aligned} \tag{6}$$

because the second term on the right side of Eq. (6), which contributes with the magnetic error θ_x, is much smaller than the firs term.

If Eqs. (5) and (6) are substituted in Eq. (3), and after performing some algebraic manipulations that involve the use of special identities [3] that reduce the sum in j, it is possible to obtain a new expression for $x(s,n)$ given by

$$\begin{aligned} x(s,n) &= \sqrt{2J_{x_0}\beta_x(s)}\Big[\sin\left(\psi_x(s)-\delta_{x_0}+2\pi Q_x(n-1)\right] \\ &+ F_t\sin(2\pi Q_x n) - \frac{nL}{2}\cos\left[\psi_x(s)-\delta_{x_0}+2\pi Q_x(n-1)\right]\Big], \end{aligned} \tag{7}$$

where F_t is defined by

$$F_t = \frac{\csc(2\pi Q_x)}{2}\left[F_s\sin\left(\psi_x+\delta_{x0}\right)+F_c\cos\left(\psi_x+\delta_{x0}\right)\right], \tag{8}$$

and F_s, F_c, and L are

$$\begin{aligned} F_s = \sum_i^m B_{1i}\beta_{xi}\sin(2\psi_{xi}),\quad F_c = \sum_i^m B_{1i}\beta_{xi}\cos(2\psi_{xi}) \\ L = \sum_i^m B_{1i}\beta_{xi} \end{aligned} \tag{9}$$

with β_{xi}=$\beta_x(s_i)$ and $\psi_{xi} = \psi_x(s_i)$ for convenience. The new form for $x(s,n)$ in Eq. (7) allows to reproduce experimental trajectories using the nominal lattice functions such as β and ψ, and the initial action and phase constants (J_{x_0}, δ_{x_0}) which can be determined as described in [1]. Although in principle the constants F_t and L are unknown, they should contain the information of the quadrupole magnetic errors.

APPROXIMATION AT SECOND ORDER FROM PERTURBATION THEORY

An improved expression for $x(s,n)$ can be obtained if the perturbation theory is used to evaluate $x_j(si)$ in Eq. (5), but this time using the result obtained in Ec. (7). This requires that the sum in j of Eq. (3) has to be expanded again. As a result of this idea and after extensive algebraic manipulations, a second form is reached for the multiturn $x(s,n)$ according to

$$\begin{aligned} x(s,n) = \sqrt{2J_{x_0}\beta_x(s)}\Big[&\sin\left(\psi_x(s)-\delta_{x_0}+2\pi Q_x(n-1)\right) \\ &+\left(As_0+As_1 n+As_2 n^2\right)\sin(2\pi Q_x n) \\ &+\left(Ac_1 n+Ac_2 n^2\cos(2\pi Q_x n)\right)\Big], \end{aligned} \tag{10}$$

where the coefficients As_0, As_1, Ac_1, As_2 and Ac_2 are functions of a set of five parameters: F_t, L and three others similar to them. Since these functions As_k and Ac_k involve products with each other of such parameters, then it is possible to neglect those second order terms due to these are much smaller than the unit, according to the reasoning done in the previous section. With this in mind, the resulting expressions for As_k and Ac_k are described by

$$\begin{aligned} As_0 &= F_t, \qquad As_1 = \frac{L}{2}\sin\left(\psi_x-2\pi Q_x-\delta_{x0}\right), \\ Ac_1 &= \frac{L}{2}\cos\left(\psi_x-2\pi Q_x-\delta_{x0}\right), \\ As_2 &= -\frac{L^2}{8}\cos\left(\psi_x-2\pi Q_x-\delta_{x0}\right), \\ Ac_2 &= -\frac{L^2}{8}\sin\left(\psi_x-2\pi Q_x-\delta_{x0}\right). \end{aligned} \tag{11}$$

From this result it is clear that coefficients As and Ac, and therefore the parameters F_t and L, can be estimated by fitting the experimental data to Eq. (10). In particular Ft and L can be calculated with high accuracy as shown in the next section.

On the other hand, an interesting consequence of this proposal is that it is possible to determine approximately the canonically conjugate variable of $x(s,n)$, that is, $x'(s,n)$, simply by taking the derivative of Eq. (10) with respect to the position coordinate s. Then, it can be shown that

$$\begin{aligned} x'(s,n) = &-\frac{\alpha_x(s)}{\beta_x(s)}x(s,n) \\ &+\sqrt{\frac{2J_{x_0}}{\beta_x(s)}}\Big[\cos(\psi_x(s)+2\pi Q_x(n-1)-\delta_{x_0}) \\ &+\left(Bs_0+Bs_1 n+Bs_2 n^2\right)\sin(2\pi Q_x n) \\ &+\left(Bc_0+Bc_1 n+Bc_2 n^2\right)\cos(2\pi Q_x n)\Big], \end{aligned} \tag{12}$$

with

$$Bs_k = \beta_x(s)\frac{dAs_k(s)}{ds},\quad Bc_k = \beta_x(s)\frac{dAc_k(s)}{ds}. \tag{13}$$

As with $x(s,n)$, $x\prime(s,n)$ is also a function that only depends on the nominal lattice functions and the coefficients from

ISBN: 978-3-95450-223-3 ISSN: 2673-7000 doi:10.18429/JACoW-NAPAC2019-WEPLS03

Eq. (11), as observed in Eqs. (12) and (13). The benefit of having $x(s,n)$ and $x'(s,n)$ simultaneously for a position s and a specific turn n implies that it is possible to calculate the action and phase after the quadrupole error using the collected information in a single BPM, as can be read from

$$\begin{aligned}\delta_{x_1} &= \psi_x(s) - \tan^{-1}\left[\frac{x(s,n)}{\alpha_x(s)x(s,n)+\beta_x(s)x'(s,n)}\right] \\ J_{x_1} &= \frac{x^2(s,n)}{2\beta_x(s)\sin^2\left[\psi_x(s)-\delta_{x_1}\right]}\end{aligned} \tag{14}$$

COMPARISON OF ANALYTICAL MODEL WITH SIMULATED DATA

Simulated LHC trajectories are generated with MADX [4], which is able to simulate turn by turn trajectories. Magnetic errors are simulated by setting to a certain strength a normal quadrupole at $s_\theta = 21747$ m where a BPM of an arc is installed.

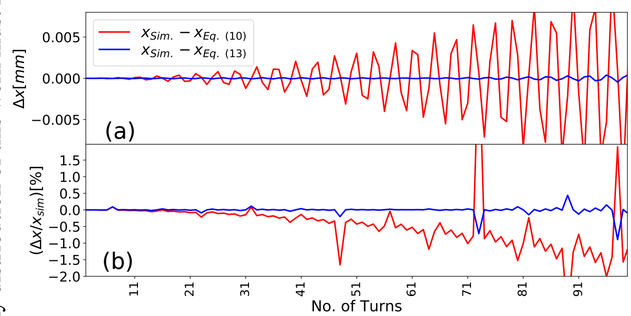

Figure 1: (a) Difference between simulated multi-turn trajectories with the first and second order analytical expressions from Eqs. (7) (red) and (10) (blue), respectively. (b) Relative difference between simulated data and analytical expression for the both same cases in (a).

Figure 1 shows the difference $\Delta x(s,n)$ between simulation and the analytical expressions for $x(s,n)$ in Eqs. (7) and (10). In both cases the discrepancy of the model with respect to the simulation is less than 1% on average, at least for the first 50 turns. But the expression given by (13) is especially close to the simulation as can be seen from the blue curve, with a relative difference less than 0.5%. Therefore, a very large number of turns is not required to determine with certain accuracy the values of $x'(s,n)$, and finally estimate the values of J_{z_1} and δ_{z_1}.

In addition, the values of F_t and L were calculated from Eq. (11) by fitting the simulated data to the expression (10). These have been compared with theoretical values given in Eqs. (8) and (9), obtaining relative differences around 0.1% and 0.04%, respectively.

Finally, Table 1 shows the relative difference between the obtained J_{x_1} and δ_{x_1} from Eq. (14) and the same quantities obtained through the usual APJ technique with two BPMS, for different turns. The agreement reached between both methods demonstrates that it would be possible to use a single BPM to calculate the action and phase constants, especially in the IRs where the number of BPMs is limited.

Table 1: Relative differences of the action and phase constants obtained from Ec. (17) and using Eq. (2) from [2].

No. Turn	$\Delta J_{x_1}/J_{x_1}$ [%]	$\Delta\delta_{x_1}/\delta_{x_1}$ [%]
5	0.01	0.0005
15	0.01	0.004
50	0.02	0.002

CONCLUSION

An alternative analytical expression has been proposed to describe a multiturn trajectory $x(s,n)$ at a point after the position of a quadrupole error using perturbation theory. From this result it is also possible to calculate the canonically conjugate variable $x'(s,n)$. With this available information, the action and phase constants were obtained and compared with the same quantities obtained from the APJ technique which uses two BPMs, achieving a good agreement.

REFERENCES

[1] J. F. Cardona and S. G. Peggs, "Linear and nonlinear magnetic error measurements using action and phase jump analysis", *Phy. Rev. ST. Accel. Beams.*, vol. 12, no. 2, p. 014002, Jan. 2009. doi:10.1103/PhysRevSTAB.12.014002

[2] J. F. Cardona and S. G. Peggs, "Erratum: Linear and nonlinear magnetic error measurements using action and phase jump analysis", *Phy. Rev. ST. Accel. Beams.*, vol. 13, no. 5, p. 059901, Jan. 2010. doi:10.1103/PhysRevSTAB.13.059901

[3] I. S. Gradshteyn and I. M. Ryzhik, "Elementary Functions", in *Tables of Integrals, Series and Products*, Academic Press, 2007, pp. 36–38.

[4] CERN – BE/ABP, Accelerator Beam Physics Group, Methodical Accelerator Design, http://mad.web.cern.ch/mad/