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ERROR MINIMIZATION IN TRANSVERSE PHASE-SPACE MEASUREMENTS USING QUADRUPOLE AND SOLENOID SCANS*

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Abstract

Quadrupole and solenoid scans are common techniques where a series of beam profile measurements are taken under varying excitation of the linear focusing elements to unfold second-order phase-space moments of the beam at an upstream location. Accurate knowledge of the moments is crucial to machine tuning and understanding the underlying beam dynamics. The scans have many sources of errors including measurement errors, field errors and misalignments. The impact of these uncertainties on the moment measurement is often not analyzed. This study proposes a scheme motivated by linear algebra error bounds that can efficiently select a set of scan parameters to minimize the errors in measured initial moments. The results are verified via a statistical error analysis. These techniques are being applied at the Facility for Rare Isotope Beams (FRIB). We find that errors in initial moments can be large under conventional scans but are greatly reduced using the procedures described.

INTRODUCTION

Quadrupole and solenoid scans (Q-scans and S-scans) are standard techniques for measuring a beam's transverse phase-space moments. The moments at an initial location are obtained by varying the strengths of focusing elements and making multiple spatial profile measurements at a downstream location, typically with wire scanners. With the assumption of linear single-particle dynamics, which is often a good approximation over a short transport length, transverse phase-space moments can be solved via a system of linear equations: $A\mathbf{x} = \mathbf{b}$ where $\mathbf{b}$ consists of measurement results, $\mathbf{x}$ are the unknown initial beam moments, and A is the coefficient matrix derived from the linear transfer map between the measurement and reconstruction points.

Transverse phase-space moments obtained by Q-scans and S-scans are subject to many sources of errors. Errors in profile measurements correspond to errors in $\mathbf{b}$, whereas errors in the matrix A arise come from errors in the linear transfer map which have many potential causes including mechanical misalignments and field errors. This study first introduces how the errors in transverse phase-space moments can be quantified. Next, we discuss how such errors can be minimized via a suitable set of scanning parameters and present an efficient method for their selection. The method is illustrated by Q-scans performed at the FRIB [1] front end. Lastly, we conclude with an outlook for further work.

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ERROR QUANTIFICATION

The errors of Q-scans and S-scans can be quantified in two ways: 1) *Statistical Analysis*; and 2) *Sensitivity Analysis*. These two approaches provide complementary information.

Statistical Analysis

To perform statistical error analysis, one repeatedly solves the equation:

$$(A + \delta A)\,(\mathbf{x} + \delta\mathbf{x}) = (\mathbf{b} + \delta\mathbf{b}) \tag{1}$$

where $\mathbf{x}$ is the unperturbed solution to $A\mathbf{x} = \mathbf{b}$, $\delta\mathbf{x}$ is the change to the unperturbed solution, and $\delta\mathbf{b}$ and δA are random perturbations to measurement results $\mathbf{b}$ and the coefficient matrix A respectively. The magnitudes of $\delta\mathbf{b}$ and δA are determined by estimates from all sources of errors. The resulting set of perturbed solutions $(\mathbf{x} + \delta\mathbf{x})$ can be plotted in histograms whose corresponding distributions can be interpreted as the probability distributions of the unknowns. As opposed to the sensitivity analysis discussed below, this treatment allows one to obtain detailed information on the error distribution.

Sensitivity Analysis

Instead of calculating the error bars on the solutions explicitly, one can employ techniques from linear algebra to analyze how sensitive the linear system of equations $A\mathbf{x} = \mathbf{b}$ is to perturbation terms δA and $\delta\mathbf{b}$. The key parameter that measures sensitivity is the condition number of A denoted by $\kappa(A)$ where:

$$\kappa(A) = \frac{\sigma_{\max}(A)}{\sigma_{\min(A)}} \tag{2}$$

with $\sigma_{\max}(A)$ and $\sigma_{\min}(A)$ being the largest and smallest non-zero singular value A respectively. We refer the reader to Ref. [2, 3] for details. Upon linearizing Eq. (1) in δ, an expression can be derived to bound the relative error of the the solution given the relative error in A and $\mathbf{b}$:

$$\begin{aligned}\frac{\|\delta\mathbf{x}\|_2}{\|\mathbf{x}\|_2} \leq{}& \kappa(A)^2 \frac{\|\mathbf{r}\|_2}{\|A\mathbf{x}\|_2}\frac{\|\delta A\|_2}{\|A\|_2} \\ &+ \kappa(A)\left(\frac{\|\mathbf{b}\|_2}{\|A\mathbf{x}\|_2}\frac{\|\delta\mathbf{b}\|_2}{\|\mathbf{b}\|_2} + \frac{\|\delta A\|_2}{\|A\|_2}\right)\end{aligned} \tag{3}$$

where $\mathbf{r} \equiv A\mathbf{x} - \mathbf{b}$ is the residual vector, and $\|.\|_2$ denotes the Euclidean norm (i.e. L_2 norm) and the associated induced norm for vectors and matrices respectively.

Given the dependence of relative errors in $\mathbf{x}$ on $\kappa(A)$ in Eq. (3), one can use $\kappa(A)$ to compare which system's solutions will have a sharper probability distribution. This

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can be done without explicitly calculating the error values and, assuming $\|\delta A\|_2 / \|A\|_2$ and $\|\delta \mathbf{b}\|_2 / \|\mathbf{b}\|_2$ are constant at different scan parameters, without even knowing the magnitudes of the error sources.

ERROR MINIMIZATION IN Q-SCANS

Since the condition number $\kappa(A)$ is a parameter that quantifies errors, the question becomes how one can set quadrupole scan parameters such that the $\kappa(A)$ is small in the resulting system of linear equations $A\mathbf{x} = \mathbf{b}$. Using the condition number to guide error minimization of beam measurements is inspired by the work at the GSI Helmholtz Centre for Heavy Ion Research (GSI) [4, 5]. However, GSI studies did not appear to analyze how to choose scan parameters such that the condition number is minimized.

A simple estimate shows that an exhaustive search over all possible sets of scan parameters for the minimum $\kappa(A)$ is impractical. Suppose there are two "knobs" (e.g. focusing strengths in a quadrupole doublet) where each knob can attain 10 values, thus giving 100 possible settings in total. To choose a set of scan parameters for four measurements, the number of possible combinations equal:

$$\binom{100}{4} = \frac{100!}{4! \times 96!} \approx 4 \times 10^6.$$

For scans with more knobs and more measurements, the number can be orders of magnitude larger. It would be very computationally inefficient to build all possible coefficient matrices A, compute their singular values, and select the one with the smallest condition number $\kappa(A)$.

Therefore, one has to rely on other ideas to efficiently obtain a set of scan parameters. A group at the Paul Scherrer Institute (PSI) proposed choosing quadrupole parameters that correspond to discrete steps in particle phase advance between the measurement and reconstruction point [6]. However, the phase advance describes rotation in normal coordinates, whereas the actual rotation in phase-space depends on the orientation of the invariant Courant-Snyder ellipse, which is different for each focusing setting.

Projection Angle

We believe that a viewpoint in terms of projection is direct and beneficial. Quadrupole settings alter the linear map from z_i to z_f and in effect provide a different projection of the initial phase-space onto the 1D spatial measurements of the beam profile monitor. If we choose a set of measurements which correspond to a diverse range of projection angles, the corresponding matrix A should have a low condition number which reduces uncertainties in the solution.

The projection angle corresponding to the x-plane linear map of a quadrupole transport line can be found as follows. The initial(i) and final (f) phase-space coordinates are related by:

$$\begin{pmatrix} x_f \\ x'_f \end{pmatrix} = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} \begin{pmatrix} x_i \\ x'_i \end{pmatrix} \tag{4}$$

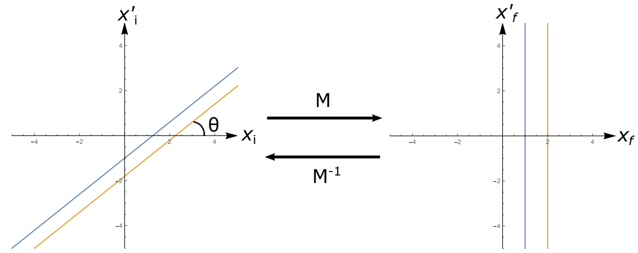

Figure 1: Linear map transforms a vertical strip in x_f-x'_f phase-space into a slanted strip in x_i-x'_i phase-space, thus illustrating how the final spatial profile is a projection of the initial phase-space distribution from an angle θ.

Therefore, the line $x_f = a_0$ in x_f-x'_f phase-space, where a_0 is a constant, becomes:

$$m_{11}x_i + m_{12}x'_i = a_0 \tag{5}$$

whose slope is given by:

$$\tan\theta = -\frac{m_{11}}{m_{12}} \tag{6}$$

Hence θ corresponds to the projection angle on the initial phase-space. The argument is illustrated by Fig. 1. Here we employ normalized dimensionless coordinates in measuring x and x' (e.g. normalization by 1 mm and 1 mrad respectively).

FRIB FRONT END EXAMPLE

To test the implementation of the methods developed, two quadrupole scans were conducted using the first profile monitor at the FRIB Front End with a 12 keV/u Ar^{9+} beam. A schematic of the relevant beam line section is shown in Fig. 2. There are two electrostatic quadrupole doublets between the measurement and reconstruction points. The voltages (+ denotes focusing in the x-plane) applied to the four quadrupoles are listed in Table 1. The parameters in Scan 1 is typical of how quadrupole scans were usually performed where only the focusing strength of the quadrupole immediately upstream of the profile monitor is varied. The parameters in Scan 2 were chosen using the techniques described in the section above to sample the initial distribution from a wide range of projection angles. The projection angles on the initial phase-space corresponding to each scan are shown in Fig. 3. Note that Scan 2 has a much wider spread of angles than Scan 1.

The performance of Scan 1 and Scan 2 in beam moment measurements are compared. phase-space moments measured at an upstream Allison scanner is propagated to the reconstruction location to serve as a benchmark. Hard edge equivalent transfer matrices are applied for the quadrupoles. Beam moments are calculated from a system of linear equations where the coefficient matrix A depends on the choice of scan parameters. In the current example, the condition number $\kappa(A)$ as defined in in Eq. (2) equals 466 and 14.5 for Scan 1 and Scan 2 respectively. With this $>$ 30-fold

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Table 1: Quadrupole Parameters for Scan 1 and Scan 2

	Scan 1						Scan 2				
Measurement	1	2	3	4	5	6	1	2	3	4	5
V_1 (V)	-2657	-2657	-2657	-2657	-2657	-2657	-500	-1500	-4000	-4000	-4500
V_2 (V)	4513	4513	4513	4513	4513	4513	3000	4000	4500	5500	4500
V_3 (V)	-4295	-4295	-4295	-4295	-4295	-4295	-4500	-4500	-4500	-3000	-4500
V_4 (V)	300	1300	2300	3300	4300	5300	4000	4000	4000	500	3500

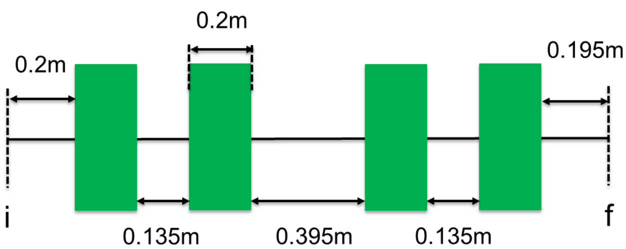

Figure 2: Beam line section containing the first profile monitor at the FRIB Front End. The profile monitor is located at the position designated by f and the beam was reconstructed at position i. Green blocks denote identical (Q7 type) electrostatic quadrupoles.

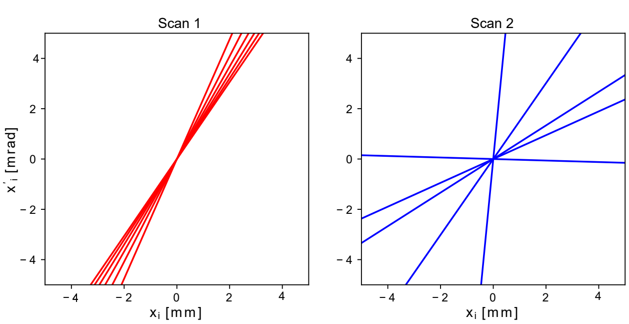

Figure 3: Projection angles on the initial phase-space corresponding to quadrupole scan parameters in Table 1.

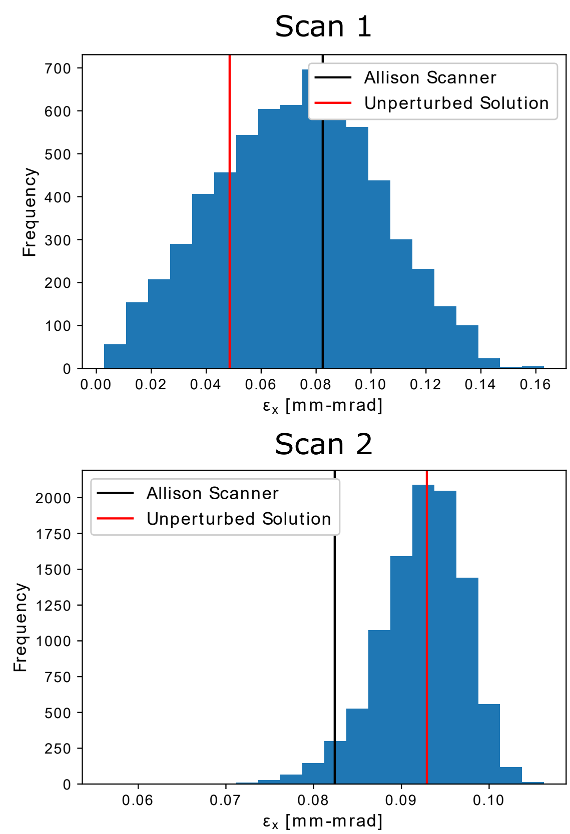

Figure 4: Histograms of the normalized rms x-emittance ε_x with random errors applied to the measurement results. The applied errors have a truncated Gaussian distribution where $3\sigma = 1\%$ for Scan 1 and $3\sigma = 10\%$ for Scan 2.

difference in the condition number, the errors in Scan 1 are expected to be much larger than those in Scan 2. This is verified by applying random errors to measurement results and solving for the emittance in each case. Histograms corresponding to 10000 perturbed solutions in each case (with larger applied errors in Scan 2) are plotted in Fig. 4, where the standard deviation of normalized rms x-emittance ε_x is 0.029 mm-mrad and 0.005 mm-mrad for Scan 1 and Scan 2 respectively. Despite the fact that applied errors for Scan 1 are 10× smaller than those for Scan 2, the measurement errors in Scan 1 are 6× larger. This demonstrates a diverse choice of projection angles is a viable methodology for reducing quadrupole scan errors.

CONCLUSION

Errors in Q-scans can be minimized by lowering the condition number $\kappa(A)$ in the system of linear equations $A\mathbf{x} = \mathbf{b}$. Qualitative methodology based on projection angles has been proposed to minimize $\kappa(A)$ and preliminary studies at the FRIB front end confirmed its efficacy. The method is being applied to develop an application for optimized automated Q-scan parameter selection at FRIB, results will be reported in the future. More rigorous arguments on which choice of coordinate system is optimal and how evenly distributed projection angles ensure low sensitivity to errors, as well as how the method can be extended to 4D phase-space, will be presented in an upcoming paper.

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REFERENCES

[1] J. Wei *et al.*, "The FRIB Superconducting Linac - status and plans," in *Proc. 28th Linear Accelerator Conf. (LINAC'16)*, East Lansing, MI, USA, Sep. 2016, pp. 1–6.
[2] J. W. Demmel, *Applied numerical linear algebra*. Siam, 1997.
[3] L. N. Trefethen and D. Bau III, *Numerical linear algebra*. Siam, 1997.
[4] C. Xiao *et al.*, "Measurement of the transverse four-dimensional beam rms-emittance of an intense uranium beam at 11.4 MeV/u," *Nucl. Instrum. Methods Phys. Res., Sect. A*, vol. 820, pp. 14–22, Jun. 2016.
[5] C. Xiao *et al.*, "Rotating system for four-dimensional transverse rms-emittance measurements," *Phys. Rev. Accel. Beams*, vol. 19, p. 072 802, Jul. 2016.
[6] E. Prat and M. Aiba, "Four-dimensional transverse beam matrix measurement using the multiple-quadrupole scan technique," *Phys. Rev. Spec .Top. Accel. Beams*, vol. 17, p. 052 801, May 2014.