# ANALYSIS AND SIMULATION OF THE ENHANCEMENT OF THE CSR EFFECTS

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*Abstract*

Recent measurements of the coherent synchrotron radiation (CSR) effects indicated that the observed beam emittance growth and energy modulation are often bigger than previous predictions based on Gaussian longitudinal charge distributions. In this paper, by performing a model study, we show both analytically and numerically that when the longitudinal bunch charge distribution involves concentration of charges in a small fraction of the bunch length, enhancement of the CSR self-interaction beyond the Gaussian prediction may occur. The level of this enhancement is sensitive to the level of the local charge concentration.


## 1 INTRODUCTION

When a short bunch with high charge is transported through a magnetic bending system, orbit-curvature-induced bunch self-interaction via CSR and space charge can potentially induce energy modulation in the bunch and cause emittance growth. Even though the earlier analytical results for CSR self-interaction [1, 2] based on the rigid-line-charge model can be applied for general longitudinal charge distributions, since the analytical results for a Gaussian beam are explicitly given, one usually applies the Gaussian results to predict the CSR effects using the measured or simulated rms bunch length. Similarly, a self-consistent simulation [3] was developed ealier to study the CSR effect on bunch dynamics for general bunch distributions; however, the simulation is usually carried out using an assumed initial Gaussian longitudinal phase space distribution. Recent CSR experiments [4, 5] indicated that the measured energy spread and emittance growth are sometimes bigger than previous Gaussian predictions, especially when a bunch is fully compressed or over-compressed. In this paper, we explore the possible enhancement of the CSR self-interaction force due to extra longitudinal concentration of charges as opposed to a Gaussian distribution. This study reveals a general feature of the CSR self-interaction: whenever there is longitudinal charge concentration in a small fraction of a bunch length, enhancement of the CSR effect beyond the Gaussian prediction can occur; moreover, the level of this enhancement is sensitive to the level of the local charge concentration within a bunch. This sensitivity should be given serious considertation in designs of future machines.

## 2 BUNCH COMPRESSION OPTICS

When an electron bunch is fully compressed by a magnetic chicane, the final bunch length and the inner structure of the final longitudinal phase space are determined by many details of the machine design. In this paper, we investigate only the RF curvature effect, which serves as a model to illustrate the possible sensitivity of the CSR interaction to the longitudinal charge distribution.

In order to study the CSR self-interaction for a compressed bunch, let us first find the longitudinal charge distribution for our model bunch when it is fully compressed by a chicane. Consider an electron bunch with $N$ total electrons. The longitudinal charge density of the bunch at time $t$ is $\rho(s,t) = Nen(s,t)$ ($\int n(s,t)ds = 1$), where $s$ is the distance from the reference electron, and $n(s,t)$ is the longitudinal density distribution of the bunch. At $t = t_0$, let the bunch be aligned on the design orbit at the entrance of a bunch compression chicane, with a Gaussian longitudinal density distribution and rms bunch length $\sigma_{s0}$

$$n(s_0,t_0) = n_0(\mu) = \frac{1}{\sqrt{2\pi}\sigma_{s0}} e^{-\mu^2/2\sigma_{s0}^2}. \qquad (1)$$

Here we let each electron be identified by the parameter $\mu$, which is its initial longitudinal position

$$s(\mu,t_0) = s_0 = \mu \qquad (s > 0 \text{ for bunch head}). \qquad (2)$$

In order to compress the bunch using the chicane, a linear energy correlation was imposed on the bunch by an upstream RF cavity, along with a slight second-order energy correlation due to the curvature of the RF wave form. The relative energy deviation from the design energy is then

$$\delta(\mu,t_0) = -\delta_1 \frac{\mu}{\sigma_{s0}} - \delta_2 \left(\frac{\mu}{\sigma_{s0}}\right)^2 (\delta_1, \delta_2 > 0, \delta_2 \ll \delta_1), \qquad (3)$$

where we assume no uncorrelated energy spread. When the beam propagates to the end of the chicane at $t = t_f$, the final longitudinal coordinates of the electrons are

$$s(\mu,t_f) = s(\mu,t_0) + R_{56}\delta(\mu,t_0) + T_{566}[\delta(\mu,t_0)]^2 \qquad (4)$$

$$= s_(\mu,t_0)(1 - \frac{R_{56}\delta_1}{\sigma_{s0}}) - \alpha[s_(\mu,t_0)]^2 \qquad (5)$$

with $\alpha \equiv (R_{56}\delta_2 - T_{566}\delta_1^2)/\sigma_{s0}^2$. One can obtain the maximum compression of the bunch by choosing the initial bunch length and the initial energy spread to satisfy

$$1 - R_{56}\delta_1/\sigma_{s0} = 0, \qquad s(\mu,t_f) = s_f = -\alpha[s(\mu,t_0)]^2. \qquad (6)$$

For typical bunch compression chicane, one has $R_{56} > 0$ and $T_{566} < 0$. Therefore $\alpha > 0$, which implies $s_f \leq 0$ from Eq. (6). Using Eqs. (6) and (2), we have

$$\mu(s_f) = \sqrt{-s_f/\alpha} \qquad (\alpha > 0, s_f \leq 0). \qquad (7)$$

The final longitudinal density distribution can be obtained from charge conservation $n_0(\mu)d\mu = n(s_f,t_f)ds_f$:

$$n(s_f,t_f) = \frac{1}{\sqrt{2\pi}\sigma_{sf}} \frac{e^{s_f/\sqrt{2}\sigma_{sf}}}{\sqrt{-s_f/\sqrt{2}\sigma_{sf}}} H(-s_f), \qquad (8)$$

$$\sigma_{sf} = \sqrt{\langle s_f^2 \rangle - \langle s_f \rangle^2} = \sqrt{2}\alpha\sigma_{s0}^2. \tag{9}$$

where $H(-s_f)$ is the Heaviside step function, and $\sigma_{sf}$ is the rms of the final longitudinal distribution. The final longitudinal phase space distribution can be obtained as

$$s_f \simeq -(\sigma_{sf}/\sqrt{2}\delta_1^2)\delta^2 \tag{10}$$

For example, when $\sigma_{s0} = 1.26$ mm, $R_{56} = 45$ mm, and $\delta_1 = 0.028$, the compression condition Eq. (6) is satisfied. With $\alpha = 0.08$ mm$^{-1}$, Eq. (9) gives the final compressed bunch length $\sigma_{sf} = 0.18$ mm. For a realistic beam, uncorrelated energy spread $\delta_{\rm un}$ should be added to Eq. (3) (here we assume $\delta_{\rm un}$ has a Gaussian distribution with $\langle\delta_{\rm un}\rangle = 0$, and rms width $\delta_{\rm un}^{\rm rms}$). As a result, one finds the final rms bunch length satisfies

$$\sigma_s^{\rm eff} = \sqrt{\langle s_f^2 \rangle - \langle s_f \rangle^2} = \sigma_{sf}\sqrt{1+a^2}, \tag{11}$$

with $\sigma_{sf}$ given by Eq. (9), and $a = R_{56}\delta_{\rm un}/\sigma_{sf}$. An example of the longitudinal phase space distribution described by Eq. (10), with an additional width due to $\delta_{\rm un} \neq 0$ as given by Eq. (11), is shown in Fig.1.

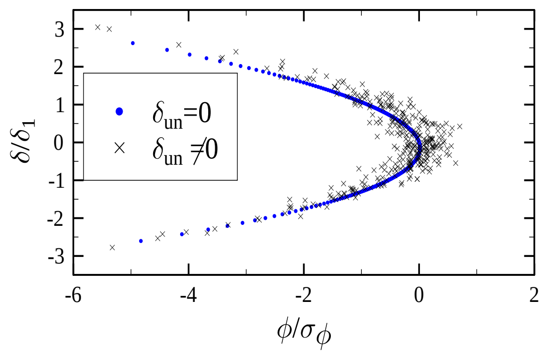


Figure 1: Example of the longitudinal phase space distribution for a compressed beam with RF curvature effect.

## 3 CSR FOR A COMPRESSED BEAM

Next, we study the CSR self-interaction of a rigid-line compressed bunch in the steady-state circular motion. The longitudinal density distribution function of the bunch is $\lambda(\phi)$ for $\phi = s/R$, with the rms angular width $\sigma_\phi = \sigma_s/R$ for the rms bunch length $\sigma_s$ and the orbit radius $R$.

### *3.1 General Formulas*

The longitudinal collective force on the bunch via space-charge and CSR self-interaction is [1, 2]:

$$\begin{aligned} F_\theta(\phi) &= \frac{e\partial(\Phi - \boldsymbol{\beta}\cdot\mathbf{A})}{\beta c\partial t} \\ &= \frac{-Ne^2}{R^2}\frac{\partial}{\partial\phi}\int_0^\infty \frac{1-\beta^2\cos\theta}{2\sin(\theta/2)}\lambda(\phi-\varphi)d\varphi \end{aligned} \tag{12}$$

where $\boldsymbol{\beta} = \mathbf{v}/c$, $\beta = |\boldsymbol{\beta}|$, $\gamma = 1/\sqrt{1-\beta^2}$, and $\theta$ is an implicit function of $\varphi$ via the retardation relation $\varphi = \theta - 2\beta\sin(\theta/2)$. In this paper, we treat only the high-energy case when $\gamma \gg \theta^{-1}$ and $\theta \simeq 2(3\varphi)^{1/3}$. In this case $F_\theta(\phi)$ is dominated by the radiation interaction:

$$F_\theta(\phi) \simeq \frac{-2Ne^2}{3^{1/3}R^2}\int_0^\infty \varphi^{-1/3}\frac{\partial}{\partial\phi}\lambda(\phi-\varphi)d\varphi. \tag{13}$$

The CSR power due to the radiation interaction is

$$P = -N\int F_\theta(\phi)\lambda(\phi)d\phi. \tag{14}$$

Results for the longitudinal collective force and the CSR power for a rigid-line Gaussian bunch are [1, 2]:

$$\lambda^{\rm Gauss}(\phi) = \frac{1}{\sqrt{2\pi}\sigma_\phi}e^{-\phi^2/2\sigma_\phi^2} \quad (\sigma_\phi \gg \frac{1}{\gamma^3}), \tag{15}$$

$$F_\theta^{\rm Gauss}(\phi) \simeq F_g g(\phi),\; F_g = \frac{2Ne^2}{3^{1/3}\sqrt{2\pi}R^2\sigma_\phi^{4/3}}, \tag{16}$$

$$P^{\rm Gauss} \simeq \frac{N^2e^2}{R^2\sigma_\phi^{4/3}}\frac{3^{1/6}\Gamma^2(2/3)}{2\pi}, \tag{17}$$

where $\Gamma(x)$ is the Gamma function, and

$$g(\phi) = \int_0^\infty \frac{(\phi/\sigma_\phi - \phi_1)}{\phi_1^{1/3}}e^{-(\phi/\sigma_\phi-\phi_1)^2/2}d\phi_1. \tag{18}$$

### *3.2 CSR Interaction for a Compressed Bunch*

The angular distribution for a compressed bunch $\lambda^{\rm cmpr}(\phi)$ with intrinsic width due to $\delta_{\rm un} \neq 0$ is the convolution of the angular density distribution $\lambda_0^{\rm cmpr}(\phi)$ with $\delta_{\rm un} = 0$ and a Gaussian distribution $\lambda_m(\phi)$:

$$\lambda^{\rm cmpr}(\phi) = \int_{-\infty}^\infty \lambda_0^{\rm cmpr}(\phi-\varphi)\lambda_m(\varphi)d\varphi, \tag{19}$$

$$\lambda_0^{\rm cmpr}(\phi) = \frac{1}{\sqrt{2\pi}\sigma_\phi}\frac{e^{\phi/\sqrt{2}\sigma_\phi}}{\sqrt{-\phi/\sqrt{2}\sigma_\phi}}H(-\phi), \tag{20}$$

$$\lambda_m(\phi) = \frac{1}{\sqrt{2\pi}\sigma_{m\phi}}e^{-\phi^2/2\sigma_{m\phi}^2},\; \sigma_{m\phi} = \frac{R_{56}\delta_{\rm un}^{\rm rms}}{R}, \tag{21}$$

where $\lambda_0^{\rm cmpr}(\phi)$ is obtained from Eq. (8). We then proceed to analyze the longitudinal CSR self-interaction force for a rigid-line bunch with the density function given in Eq. (19) under the condition $\sigma_\phi > \sigma_{m\phi} \gg \gamma^{-3}$. Combining Eq. (19) with Eq. (13), and denoting $a$ as the intrinsic width of the bunch relative to the rms bunch length $(0 < a < 1)$:

$$a = \frac{\sigma_w}{\sigma_s} \quad (\sigma_w = R_{56}\delta_{\rm un}^{\rm rms}), \tag{22}$$

one finds the steady-state CSR longitudinal force for a compressed bunch:

$$F_\theta^{\rm cmpr}(\phi) = \int_{-\infty}^\infty F_{\theta 0}^{\rm cmpr}(\varphi)\lambda_m(\phi-\varphi)d\varphi. \tag{23}$$

It can be shown that $F_{\theta 0}^{\rm cmpr}(\varphi)$ in Eq. (23) is

$$F_{\theta 0}^{\rm cmpr}(\phi) \simeq \frac{-2Ne^2}{3^{1/3}R^2}\int_0^\infty \varphi^{-1/3}\frac{\partial}{\partial\phi}\lambda_0^{\rm cmpr}(\phi-\varphi)\,d\varphi$$
$$= -2^{1/4}\,F_g\,dG(y)/dy \qquad (y=\phi/\sigma_\phi), \quad (24)$$

with $F_g$ given in Eq. (16), and

$$G(y) = H(-y)\,e^{-|y|/\sqrt{2}}|y|^{1/6}\,\Gamma\left(\frac{2}{3}\right)\Psi\left(\frac{2}{3},\frac{7}{6};\frac{|y|}{\sqrt{2}}\right)$$
$$+H(y)\,y^{1/6}\,\Gamma\left(\frac{1}{2}\right)\Psi\left(\frac{1}{2},\frac{7}{6};\frac{y}{\sqrt{2}}\right), \quad (25)$$

where $\Psi(a,\gamma;z)$ is the degenerate hypergeometric function

$$\Psi(\alpha,\gamma;z) = \frac{1}{\Gamma(\alpha)}\int_0^\infty e^{-zt}t^{\alpha-1}(1+t)^{\gamma-\alpha-1}dt. \quad (26)$$

As a result, we have

$$F_\theta^{\rm cmpr}(\phi) = \frac{2^{1/4}\,F_g}{\sqrt{2\pi}\,a^{5/6}}f\left(\frac{\phi}{a\,\sigma_\phi};a\right), \quad (27)$$

$$f(y;a) = \int_{-\infty}^{\infty} G(a\,x)(y-x)\,e^{-(y-x)^2/2}dx. \quad (28)$$

Similarly, the radiation power can also be obtained for the compressed bunch using Eq. (14) with $\lambda^{\rm cmpr}(\phi)$ in Eq. (19) and $F_\theta^{\rm cmpr}(\phi)$ in Eq. (27), which gives

$$\frac{P^{\rm cmpr}}{P^{\rm Gauss}} \simeq 0.75\,\frac{I(a)}{a^{5/6}}, \quad (29)$$

$$I(a) = -\int_{-\infty}^{\infty} f\left(\frac{\phi}{a\,\sigma_\phi};a\right)\,\lambda^{\rm cmpr}(\phi)d\phi. \quad (30)$$

Numerical integration shows that $|f(y;a)|_{\rm max}$ — the maximum of $|f(y;a)|$ for fixed $a$ — is insensitive to $a$ for $0<a<1$. As a result, for a compressed bunch with fixed $\sigma_\phi$, we found from Eq. (27) the amplitude of the CSR force $F_\theta^{\rm cmpr}(\phi)$ varies with $a^{-5/6}$. Therefore in contrast to the well-known scaling law $R^{-2/3}\sigma_s^{-4/3}$ for the amplitude of the longitudinal CSR force for a Gaussian beam, a bunch described by Eq. (19) has $|F_\theta^{\rm cmpr}|_{\rm max} \propto R^{-2/3}\sigma_s^{-1/2}\sigma_w^{-5/6}$ with $\sigma_w$ in Eq. (22) denoting the intrinsic width of the bunch. Likewise, for $a$=0.1, 0.2, and 0.5, we found from numerical integration that $I(a) \simeq$ 0.76, 0.90 and 1.02 respectively, and correspondingly $P^{\rm cmpr}/P^{\rm Gauss} \simeq$ 3.9, 2.6 and 1.4. This dependence of the amplitude of the CSR force and power on the intrinsic width of the bunch for a fixed rms bunch length manifests the sensitivity of the enhancement of the CSR effect on the local charge concentration in a longitudinal charge distribution.

In Figs. 2 and 3, we plot the longitudinal density function for various charge distributions with the same rms bunch lengths (except the $\sqrt{1+a^2}$ factor in Eq. (11)), and the longitudinal CSR collective forces associated with the various distributions. The amplitude of $F_\theta^{\rm cmpr}$ in Fig. 3 agrees with the $a^{-5/6}$ dependence in Eq. (27). Good agreement of the analytical result in Eq. (27) with the simulation result [3] for the CSR force along the example distribution in Fig. 1 is shown in Fig. 4.

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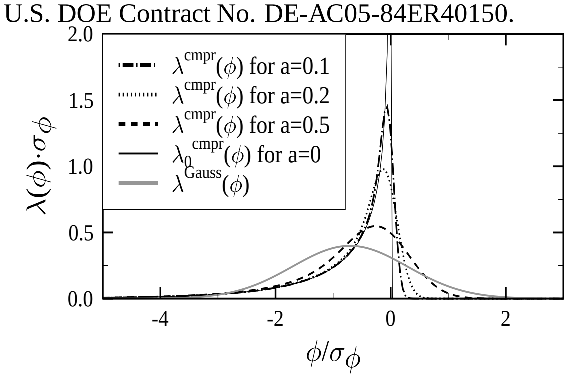


Figure 2: Longitudinal charge distribution for a compressed bunch with intrinsic width described by $a$, compared with a Gaussian distribution. All the distributions here have the same angular rms size $\sigma_\phi$.

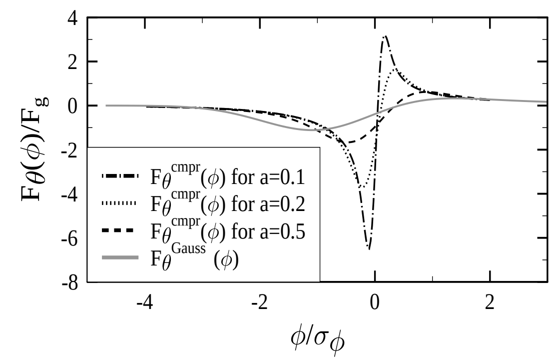


Figure 3: Longitudinal CSR force along the bunch for various charge distributions illustrated in Fig. 2.

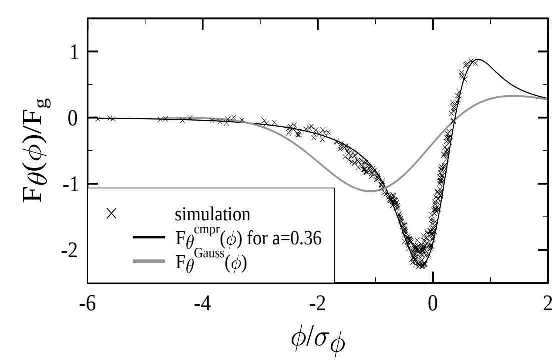


Figure 4: Comparison of the analytical and numerical results of the longitudinal CSR force along the example bunch illustrated in Fig. 1. Here we have $\sigma_x \simeq 3\sigma_s$.

## 4 REFERENCES

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