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HAMILTONIAN FORMALISM OF INTENSE BEAMS IN DRIFT-TUBE LINEAR ACCELERATORS*

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Abstract

Starting from the principle of least action, we construct a general Hamiltonian formalism for beam dynamics in drift-tube linear accelerators (DTLs). The Alvarez-type structure is considered as an example, but the present theory can readily be extended to other types of conventional linacs. The three-dimensional Hamiltonian derived here includes the third-order chromatic term as well as the effects from acceleration and space charge. A clear dynamical analogy between the DTL system and compact Paul ion-trap system is pointed out, which suggests that we can conduct a fundamental design study of high-intensity hadron linacs experimentally in a local tabletop environment instead of relying on large-scale machines.

LAGRANGIAN

The starting point is the principle of least action

$$\delta \int L_t \, dt = 0 \tag{1}$$

where L_t is the Lagrangian using time t as the independent variable. The spatial position of a charged particle traveling in the DTL can be specified by the vector $\boldsymbol{u} = z + x\boldsymbol{e}_x + y\boldsymbol{e}_y$ whose z-derivative is $\boldsymbol{u}' = \boldsymbol{e}_z + x'\boldsymbol{e}_x + y'\boldsymbol{e}_y$. Here, the prime stands for differentiation with respect to z. $(\boldsymbol{e}_x, \boldsymbol{e}_y, \boldsymbol{e}_z)$ are the unit vectors toward the transverse x-y and longitudinal z directions, perpendicular to each other. In beam dynamics, it is most convenient to take the longitudinal coordinate z, instead of t, as the independent variable. The Lagrangian of a relativistic charged particle moving under the influence of electromagnetic fields can then be given by

$$L_z \equiv L_t \frac{dt}{dz} = -mc\sqrt{(ct')^2 - \mathbf{u}' \cdot \mathbf{u}'} + q(\mathbf{A} \cdot \mathbf{u}' - \phi t'), \tag{2}$$

where m and q are the mass and charge state of the particle, c is the speed of light, and (ϕ, $\boldsymbol{A}$) are the scalar and vector potentials that satisfy the Maxwell equations. Making use of the cylindrical coordinates, we have

$$\begin{aligned} L_z = &-mc\sqrt{(ct')^2 - 1 - (r')^2 - (r\theta')^2} \\ &+ q(A_r r' + A_\theta r\theta' + A_z - \phi t'), \end{aligned} \tag{3}$$

where $\boldsymbol{A} = (A_r, A_\theta, A_z)$. The total vector potential is the sum of the contributions from the radio-frequency (rf) accelerating field ($\boldsymbol{A}^{(\mathrm{rf})}$), the transverse focusing field by quadrupole magnets ($\boldsymbol{A}^{(\mathrm{mag})}$), and the space-charge field ($\boldsymbol{A}^{(\mathrm{sc})}$); namely, $\boldsymbol{A} = \boldsymbol{A}^{(\mathrm{rf})} + \boldsymbol{A}^{(\mathrm{mag})} + \boldsymbol{A}^{(\mathrm{sc})}$. On the other hand, the scalar potential originates only from the space-charge field; namely, $\phi = \phi_{\mathrm{sc}}$.

ELECTROMAGNETIC POTENTIALS

Let us consider the regular DTL structure whose periodic length is $2\ell_{\mathrm{c}}$. Assuming the axisymmetric TM mode for particle acceleration, we obtain the following vector potential components from the Maxwell equations:

$$A_z^{(\mathrm{rf})} = -\frac{1}{\omega}\sum_{n=0}^{\infty} a_n I_0(k_n r)\cos\frac{n\pi z}{\ell_{\mathrm{c}}}\sin\omega t, \tag{4a}$$

$$A_\theta^{(\mathrm{rf})} = 0, \tag{4b}$$

$$A_r^{(\mathrm{rf})} = -\frac{1}{\omega}\sum_{n=0}^{\infty} \frac{n\pi a_n}{k_n \ell_{\mathrm{c}}} I_1(k_n r)\sin\frac{n\pi z}{\ell_{\mathrm{c}}}\sin\omega t, \tag{4c}$$

where I_n is the modified Bessel function of order n, $\omega = 2\pi c/\lambda$ with λ being the rf wavelength, a_n is the nth Fourier coefficient, and $k_n^2 = (2\pi/\lambda)^2[(n\lambda/2\ell_{\mathrm{c}})^2 - 1]$. When the velocity of the synchronous particle is $\beta_s c$, we have $\ell_{\mathrm{c}} = \beta_{\mathrm{s}}\lambda$ for the Alvarez-type DTL and $\ell_{\mathrm{c}} = \beta_{\mathrm{s}}\lambda/2$ for the Wideröe-type DTL. The following discussion focuses on the Alvarez-type structure, but the extension of the present formalism to other types of DTLs is straightforward. In order to give an approximate expression of the Fourier coefficients, we assume that the axial electric field exists only within every accelerating gap of width g and is uniform at the aperture radius r_0. Writing the axial field in each gap as $E_z(r = r_0, z) = E_0\ell_{\mathrm{c}}/g$ where E_0 is constant, we obtain

$$a_0 = \frac{E_0}{I_0(k_0 r_0)}, \tag{5a}$$

$$a_n = \frac{2E_0}{I_0(k_n r_0)} \cdot \frac{\sin(n\pi g/2\ell_{\mathrm{c}})}{n\pi g/2\ell_{\mathrm{c}}} \quad \text{for } n = 2,4,6,\cdots \tag{5b}$$

and $a_n = 0$ for odd harmonic numbers [1]. In the Alvarez DTL, the forward traveling wave of $n = 2$ is used for beam acceleration. The contribution from other traveling waves of different phase velocities can be ignored, which enables us to simplify the vector potential as

$$A_z^{(\mathrm{rf})} = \frac{E_0 T}{\omega} I_0(kr)\sin\left(\frac{2\pi z}{\ell_{\mathrm{c}}} - \omega t\right), \tag{6a}$$

$$A_r^{(\mathrm{rf})} = -\frac{2\pi E_0 T}{\omega k \ell_{\mathrm{c}}} I_1(kr)\cos\left(\frac{2\pi z}{\ell_{\mathrm{c}}} - \omega t\right), \tag{6b}$$

where $T = \sin(\pi g/\beta_{\mathrm{s}}\lambda)/(\pi g/\beta_{\mathrm{s}}\lambda)/I_0(kr_0)$, and $k \equiv k_2 = 2\pi/\beta_{\mathrm{s}}\gamma_{\mathrm{s}}\lambda$ with $\gamma_{\mathrm{s}} = 1/\sqrt{1-\beta_{\mathrm{s}}^2}$.

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The transverse components of the vector potential $\boldsymbol{A}^{(\mathrm{mag})}$ can be ignored provided that the beam focusing magnets produces no axial multipole fields; namely, $A_r^{(\mathrm{mag})} = 0 = A_\theta^{(\mathrm{mag})}$, and

$$A_z^{(\mathrm{mag})} = \sum_{n=1}^{\infty} B_n \left(\frac{r}{r_0}\right)^n \cos(n\theta + \zeta_n), \tag{7}$$

where B_n and ζ_n are constants. If we regard the beam as a uniform axial current, the space-charge-induced vector potential $\boldsymbol{A}^{(\mathrm{sc})}$ can also be assumed to have the axial component only; namely, $\boldsymbol{A}^{(\mathrm{sc})} = (0, 0, A_z^{(\mathrm{sc})})$ where $A_z^{(\mathrm{sc})}$ is related to the scalar potential as $A_z^{(\mathrm{sc})} \approx \beta_s \phi_{sc} / c$.

HAMILTONIAN

Using the electromagnetic potentials defined in the last section, we can derive the Hamiltonian from the Lagrangian in Eq. (3) as

$$\begin{aligned} H_{\mathrm{DTL}} = & -\sqrt{p^2 - (p_r - qA_r^{(\mathrm{rf})})^2 - \frac{p_\theta^2}{r^2}} \\ & - q(A_z^{(\mathrm{rf})} + A_z^{(\mathrm{mag})}) - \frac{q\beta_s}{c}\phi_{\mathrm{sc}}, \end{aligned} \tag{8}$$

where the canonical momenta conjugate to the coordinates (r, θ, t) have been denoted by (p_r, p_θ, p_t), and p represents the kinetic momentum given by $p = \sqrt{(p_t + q\phi_{\mathrm{sc}})^2 / c^2 - m^2c^2}$. For the synchronous particle, p is reduced to the design kinetic momentum

$$p_{\mathrm{s}} \equiv \sqrt{(W_{\mathrm{s}}/c)^2 - m^2c^2} = m\beta_{\mathrm{s}}\gamma_{\mathrm{s}}c, \tag{9}$$

where the total energy W_s of the synchronous particle increases according to

$$\frac{dW_{\mathrm{s}}}{dz} = -q\left(\frac{\partial A_z^{(\mathrm{rf})}}{\partial t}\right)_{\substack{r=0 \\ \psi=\psi_{\mathrm{s}}}} \approx qE_0T\cos\psi_{\mathrm{s}} \tag{10}$$

with the synchronous phase defined by

$$\psi_{\mathrm{s}} = \omega\int^z \frac{dz}{\beta_{\mathrm{s}}c} - \frac{2\pi z}{\ell_{\mathrm{c}}}. \tag{11}$$

In the present paper, ψ_s is assumed to be constant from the entrance to the exit of the DTL. The Coulomb potential ϕ_{sc} satisfies the Poisson equation while the particle distribution function in phase space obeys the Vlasov equation. The canonical equations of particle motion in the DTL can be obtained from the Hamiltonian H_{DTL} in Eq. (8) together with the external field potentials given by Eqs. (6) and (7).

We now expand the square root in Eq. (8) into a power series and retain only low-order terms under the assumption that p is much greater than the transverse momenta:

$$\begin{aligned} H_{\mathrm{DTL}} \approx & -p + \frac{1}{2p}\left[(p_r - qA_r^{(\mathrm{rf})})^2 + \frac{p_\theta^2}{r^2}\right] \\ & - q(A_z^{(\mathrm{rf})} + A_z^{(\mathrm{mag})}) - \frac{q\beta_s}{c}\phi_{\mathrm{sc}}. \end{aligned} \tag{12}$$

The momentum p can also be expanded about the synchronous value p_s as

$$\begin{aligned} p = p_{\mathrm{s}} & + \frac{\Delta W - q\phi_{\mathrm{sc}}}{\beta_{\mathrm{s}}c} - \frac{1}{2p_{\mathrm{s}}}\left(\frac{\Delta W}{\beta_{\mathrm{s}}\gamma_{\mathrm{s}}c}\right)^2 \\ & + \frac{q}{p_{\mathrm{s}}}\cdot\frac{\phi_{\mathrm{sc}}\Delta W}{(\beta_{\mathrm{s}}\gamma_{\mathrm{s}}c)^2} + \frac{\gamma_{\mathrm{s}}}{2p_{\mathrm{s}}^2}\left(\frac{\Delta W}{\beta_{\mathrm{s}}\gamma_{\mathrm{s}}c}\right)^3 + \cdots, \end{aligned} \tag{13}$$

where $-\Delta W$ is the energy deviation from the synchronous value W_s; namely, $-\Delta W = p_t + W_{\mathrm{s}}$. For later convenience, we transform the longitudinal canonical variables (t, p_t) to the relative time and energy (Δt, $-\Delta W$), employing the generating function

$$F_1(t, -\Delta W; z) = -(\Delta W + W_{\mathrm{s}})\left(t - \int \frac{dz}{\beta_{\mathrm{s}}c}\right). \tag{14}$$

The transformed Hamiltonian is

$$\begin{aligned} H_{\mathrm{DTL}} \approx & \frac{1}{2p_{\mathrm{s}}}\left(\frac{\Delta W}{\beta_{\mathrm{s}}\gamma_{\mathrm{s}}c}\right)^2 - \frac{\gamma_{\mathrm{s}}}{2p_{\mathrm{s}}}\left(\frac{\Delta W}{\beta_{\mathrm{s}}\gamma_{\mathrm{s}}c}\right)^3 \\ & + \frac{1}{2p_{\mathrm{s}}}\left(1 - \frac{\Delta W}{p_{\mathrm{s}}\beta_{\mathrm{s}}c}\right)\left[p_r + \frac{2\pi qE_0T}{\omega k\ell_{\mathrm{c}}}I_1(kr)\cos\left(\omega\Delta t + \psi_{\mathrm{s}}\right)\right]^2 \\ & + \frac{1}{2p_{\mathrm{s}}}\left(1 - \frac{\Delta W}{p_{\mathrm{s}}\beta_{\mathrm{s}}c}\right)\frac{p_\theta^2}{r^2} - q\sum_{n=1}^{\infty} B_n\left(\frac{r}{r_0}\right)^n \cos(n\theta + \zeta_n) \\ & + \frac{qE_0T}{\omega}\left[I_0(kr)\sin\left(\omega\Delta t + \psi_{\mathrm{s}}\right) - \omega\Delta t\cos\psi_{\mathrm{s}}\right] \\ & + \frac{q}{\beta_{\mathrm{s}}\gamma_{\mathrm{s}}^2c}\left(1 - \frac{\Delta W}{p_{\mathrm{s}}\beta_{\mathrm{s}}c}\right)\phi_{\mathrm{sc}}, \end{aligned} \tag{15}$$

where we have ignored the fourth and higher order terms. The nonlinear Hamiltonian in Eq. (15) includes many natural synchro-betatron coupling terms that may give rise to non-negligible dynamic effects under certain conditions. For instance, the term proportional to the third-order product $p_r^2 \cdot \Delta W$ yields the energy-dependent modulation of the transverse focusing force (the chromatic effect). The strict period of the longitudinal driving force is no longer a single cell structure but determined by the transverse magnetic lattice when the coupling is strong. The possible maximum synchrotron phase advance per cell may then be limited to avoid resonance.

LINEAR MODEL

The linear dynamics is most important in practice. Since the radial coordinate r of any particle is generally much less than the cell length ℓ_{c}, we have $kr \ll 1$, which allows us to put $I_0(kr) \approx 1 + (kr)^2/4$ and $I_1(kr) \approx kr/2$. Substituting these relations in Eq. (15) and dropping all nonlinear terms (except for the space-charge potential), we eventually reach the Hamiltonian

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$$H_{\mathrm{DTL}} \approx \frac{1}{2p_{\mathrm{s}}}\left(\frac{\Delta W}{\beta_{\mathrm{s}}\gamma_{\mathrm{s}}c}\right)^2 - \frac{qE_0T\omega\sin\psi_{\mathrm{s}}}{2}(\Delta t)^2 + \frac{1}{2p_{\mathrm{s}}}\left(p_r + \frac{\pi qE_0T}{\omega\ell_{\mathrm{c}}}r\cos\psi_{\mathrm{s}}\right)^2 + \frac{p_\theta^2}{2p_{\mathrm{s}}r^2} + \frac{qG(z)}{2}r^2\cos2\theta + \frac{qE_0T\sin\psi_{\mathrm{s}}}{4\omega}(kr)^2 + \frac{q\phi_{\mathrm{sc}}}{\beta_{\mathrm{s}}\gamma_{\mathrm{s}}^2c}, \quad (16)$$

where $-G(z)$ is the z-dependent step function representing the field gradient of the quadrupole magnet along the beam line. Equation (16) is somewhat simplified by introducing the new radial canonical variables $(\hat{r},\hat{p}_r)$ generated by

$$F_2(r,\hat{p}_r;z) = r\hat{p}_r - \frac{\pi qE_0T}{2\omega\ell_{\mathrm{c}}}r^2\cos\psi_{\mathrm{s}}. \quad (17)$$

After the transformation, the Hamiltonian takes the form

$$H_{\mathrm{DTL}} \approx \frac{(\Delta W)^2}{2p_{\mathrm{s}}(\beta_{\mathrm{s}}\gamma_{\mathrm{s}}c)^2} + \frac{1}{2p_{\mathrm{s}}}\left[\hat{p}_r^2 + \left(\frac{p_\theta}{r}\right)^2\right] + \frac{p_{\mathrm{s}}}{2}\left(\frac{\gamma_{\mathrm{s}}\sigma_\parallel}{2\pi}\right)^2(\omega\Delta t)^2 - \frac{p_{\mathrm{s}}\sigma_\parallel^2}{4}\left[1-\left(\frac{\gamma_{\mathrm{s}}\sigma_\parallel}{2\pi}\cot\psi_{\mathrm{s}}\right)^2\right]\left(\frac{r}{\ell_{\mathrm{c}}}\right)^2 + \frac{qG(z)}{2}r^2\cos2\theta + \frac{q}{\beta_{\mathrm{s}}\gamma_{\mathrm{s}}^2c}\phi_{\mathrm{sc}}, \quad (18)$$

where $\hat{r}$ has been replaced by r for brevity because $\hat{r}=r$, and $\sigma_\parallel$ is the synchrotron phase advance defined by

$$\sigma_\parallel^2 = -\frac{2\pi q\lambda E_0T\sin\psi_{\mathrm{s}}}{mc^2\beta_{\mathrm{s}}\gamma_{\mathrm{s}}^3} \quad (19)$$

for a negative synchronous phase ($\psi_{\mathrm{s}}<0$). Scaling the variables, we can further simplify the Hamiltonian to

$$\tilde{H}_{\mathrm{DTL}} = \frac{1}{2}\left[\tilde{p}_r^2 + \left(\frac{\tilde{p}_\theta}{r}\right)^2 + \tilde{p}_z^2\right] + \frac{\sigma_\parallel^2}{2}\left(\frac{\Delta z}{\ell_{\mathrm{c}}}\right)^2 + \left[\frac{qG(z)\ell_{\mathrm{c}}^2\cos2\theta}{2p_{\mathrm{s}}} - \frac{\sigma_\parallel^2}{4}\right]\left(\frac{r}{\ell_{\mathrm{c}}}\right)^2 + \frac{q}{p_{\mathrm{s}}\beta_{\mathrm{s}}\gamma_{\mathrm{s}}^2}\phi_{\mathrm{sc}}, \quad (20)$$

where the spatial coordinates are $(r,\theta,\Delta z\equiv\beta_{\mathrm{s}}\gamma_{\mathrm{s}}c\Delta t)$, the canonical momenta conjugate to these coordinates are $(\tilde{p}_r,\tilde{p}_\theta,\tilde{p}_z)$, and we have disregarded the small longitudinal tune shift due to rf acceleration.

ANALOGY WITH PAUL ION TRAP

It is possible to confine a large number of ions in a compact Paul trap using electric potentials [2]. The collective motion of an ion plasma stored in a Paul trap can be shown physically equivalent to that of a charged-particle beam traveling in an alternating-gradient transport channel [3]. Since the plasma motion is non-relativistic, the Lagrangian in the time domain can be written as

$$L_t = \frac{1}{2}m\left[\dot{r}^2 + (r\dot{\theta})^2 + \dot{z}^2\right] + q\left(A_r\dot{r} + A_\theta r\dot{\theta} + A_z\dot{z} - \phi\right), \quad (21)$$

where the dot mark stands for time derivative. In a regular Paul ion trap, no magnetic field is employed which means that we can simply put $A_r = A_\theta = A_z = 0$. The scalar potential includes the contributions from the space-charge interaction (ϕ_{sc}) and external electric fields for ion confinement (ϕ_{ext}). Then, the Hamiltonian derived from the Lagrangian in Eq. (21) takes the very simple form as

$$H_{\mathrm{LPT}} = \frac{1}{2m}\left[p_r^2 + \left(\frac{p_\theta}{r}\right)^2 + p_z^2\right] + q(\phi_{\mathrm{ext}} + \phi_{\mathrm{sc}}). \quad (22)$$

ϕ_{ext} can be divided into the transverse and longitudinal potential components, i.e., $\phi_{\mathrm{ext}} = \phi_\perp + \phi_\parallel$. For the transverse plasma focusing, we utilize the rf quadrupole potential $\phi_\perp = V_\perp(t)(r/r_0)^2\cos2\theta$ where r_0 is the radius of the trap aperture, and $V_\perp(t)$ corresponds to the amplitude of the rf voltages on the four quadrupole rods placed symmetrically around the trap axis. The axial ion confinement is achieved by applying DC or AC voltages to the separate quadrupole sections at both ends. Choosing a proper mechanical design, we can make the axial potential well approximately parabolic; namely, $\phi_\parallel \approx V_\parallel(z^2 - r^2/2)/\ell_z^2$ where $V_\parallel$ is related to the voltages on the end sections, and ℓ_z is the characteristic distance depending on the trap design. Substituting these potentials in Eq. (22) and scaling the variables, we obtain

$$\tilde{H}_{\mathrm{LPT}} = \frac{1}{2}\left[\tilde{p}_r^2 + \left(\frac{\tilde{p}_\theta}{r}\right)^2 + \tilde{p}_z^2\right] + \frac{\sigma_\parallel^2}{2}\left(\frac{z}{\ell_{\mathrm{LPT}}}\right)^2 + \left[\frac{qV_\perp(\tau)}{mc^2}\left(\frac{\ell_{\mathrm{LPT}}}{r_0}\right)^2\cos2\theta - \frac{\sigma_\parallel^2}{4}\right]\left(\frac{r}{\ell_{\mathrm{LPT}}}\right)^2 + \frac{q}{mc^2}\phi_{\mathrm{sc}}, \quad (23)$$

where $\ell_{\mathrm{LPT}} \equiv \lambda_{\mathrm{LPT}}/2$ with λ_{LPT} being the wavelength of the operating rf field, and the independent variable is $\tau = ct$. The synchrotron phase advance has been defined here as

$$\sigma_\parallel^2 = \frac{2qV_\parallel}{mc^2}\left(\frac{\ell_{\mathrm{LPT}}}{\ell_z}\right)^2. \quad (24)$$

Note that $V_\parallel$ is a time-dependent function if we use an rf voltage for axial ion confinement. Apart from the details of the coefficients, $\tilde{H}_{\mathrm{LPT}}$ in Eq. (23) is identical to $\tilde{H}_{\mathrm{DTL}}$ in Eq. (20). What happens in the former dynamical system, therefore, also happens in the latter, which indicates that the compact ion trap can be employed for the fundamental study of beam dynamics in DTLs.

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