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MATRIX APPROACH TO DECOUPLE TRANSVERSE-COUPLED BEAMS*

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Abstract

Transverse emittances, especially vertical emittance, are strictly required in the synchrotrons with multi-loop injection. Transverse emittances easily grow up if transverse beam phase spaces are coupled. The growth of the transverse emittance can be restained by decoupling the beam phase spaces. Based on the transfer matrix calculation, it can be theoretically proved that the decoupling can be implemented for general situations. A minimum number of rotated quadrupoles required for decoupling is given. Two quadrupoles can decouple the beam and suppress its emittance growth to 1% in the coupling DTL case.

INTRODUCTION

Rotated quadrupoles and solenoids can lead to the coupling of beam dynamics between two transverse phase spaces [1, 2]. The coupling will result in transverse emittance growth. The theories of describing the coupled dynamics with matrix [3, 4], Hamiltonian theory [5, 6], and Courant-Snyder theory [7, 8] have been developed.

It is important to decouple the transverse dynamics and suppress the emittance growth, especially for the beam injection of the synchrotron which has a strict transverse acceptance limit. It has been illustrated that the beam can be decoupled by rotated quadrupoles in specific situations [9, 10]. Normal triplet and skew triplet after the beamline are used to decouple the beam coupled by a solenoid [11]. Quadrupole-solenoid-quadrupole system is used to cancel the coupling in a solenoid [1]. The coupling in a solenoid is eliminated by a quadrupole corrector consisting of normal and skew quadrupoles [12].

In this paper, the implementation of decoupling for general cases regardless of the coupling sources is verified. The minimum number of rotated quadrupoles required for decoupling is discussed.

BASIC DEFINITIONS

The beam can be described in a beam matrix. The transverse matrix is symmetric with ten independent variables,

$$\Sigma = \begin{pmatrix} \langle xx\rangle & \langle xx'\rangle & \langle xy\rangle & \langle xy'\rangle \\ \langle xx'\rangle & \langle x'x'\rangle & \langle x'y\rangle & \langle x'y'\rangle \\ \langle xy\rangle & \langle x'y\rangle & \langle yy\rangle & \langle yy'\rangle \\ \langle xy'\rangle & \langle x'y'\rangle & \langle yy'\rangle & \langle y'y'\rangle \end{pmatrix} = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{yx} & \sigma_{yy} \end{pmatrix}. \tag{1}$$

where σ_{xx}, σ_{xy}, σ_{yx}, σ_{yy} are 2×2 matrices, and $\sigma_{yx} = \sigma_{xy}^T$. The four-dimensional RMS emittance ε_{4D}, RMS emittances in x plane ε_x, and RMS emittances in y plane ε_y are the square roots of the determinant of Σ, σ_{xx}, and σ_{yy}, respectively.

The elements in σ_{xy} or σ_{yx} represent the coupling in x and y planes. If either of them is nonzero, the beam is coupled in x and y planes. Thus $\varepsilon_{4D} < \varepsilon_x\varepsilon_y$, which is illustrated in the appendix. It suggests the product of the emittances of x and y planes increases after coupling. If $\sigma_{xy} = \sigma_{yx} = 0$, the beam motion is decoupled in x and y planes, and $\varepsilon_{4D} = \varepsilon_x\varepsilon_y$. The beam matrix at s_2 can be calculated by transfer matrix R from s_1 to s_2 and the beam matrix at s_1,

$$\Sigma(s_2) = R\Sigma(s_1)R^T. \tag{2}$$

R obeys the symplecticity condition,

$$S = R^T SR, \tag{3}$$

which is a congruent transformation, with

$$S = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}. \tag{4}$$

The determinant of R is 1 and the 4D phase space volume can be conserved according to the Liouville theorem if the beam is not accelerated nor bended, thus ε_{4D} is conserved. $\Sigma(s_2)$ is symplectically congruent to $\Sigma(s_1)$.

FEASIBILITY OF DECOUPLING

To solve the problem of decoupling, the existence of decoupling symplectic matrix R for general cases is theoretically proved using linear algebra techniques. The beam is decoupled by symplectically congruent transforming of the transverse beam matrix to a diagonal matrix.

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In particular, the matrix Σ is symplectically congruent to a diagonal matrix only if Σ is symmetric and $\Sigma S^{-1}\Sigma S$ is diagonalizable [13].

The beam matrix Σ is symmetric. Therefore we need to prove that $\Sigma S^{-1}\Sigma S$ is diagonalizable.

Considering $S^{-1} = S^T = -S$, we can obtain

$$\Sigma S^{-1}\Sigma S = -\Sigma S\Sigma S = \begin{pmatrix} A_{xx} & A_{xy} \\ A_{yx} & A_{yy} \end{pmatrix}, \tag{5}$$

where

$$A_{xx} = \left[\det(\sigma_{xx}) + \det(\sigma_{xy})\right] I_2, \tag{6a}$$

$$A_{yy} = \left[\det(\sigma_{yy}) + \det(\sigma_{xy})\right] I_2, \tag{6b}$$

$$A_{xy} = -(\sigma_{xx}J\sigma_{xy}J + \sigma_{xy}J\sigma_{yy}J), \tag{6c}$$

$$A_{yx} = -(\sigma_{yx}J\sigma_{xx}J + \sigma_{yy}J\sigma_{yx}J), \tag{6d}$$

I_n is the n-d identity matrix, and $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

The eigenvalues of $\Sigma S^{-1}\Sigma S$ can be solved as

$$\lambda_1 = \lambda_2 = \frac{1}{4}\left[-\mathrm{tr}(\Sigma S\Sigma S) - \sqrt{(\mathrm{tr}(\Sigma S\Sigma S))^2 - 16\det(\Sigma)}\right], \tag{7a}$$

$$\lambda_3 = \lambda_4 = \frac{1}{4}\left[-\mathrm{tr}(\Sigma S\Sigma S) + \sqrt{(\mathrm{tr}(\Sigma S\Sigma S))^2 - 16\det(\Sigma)}\right], \tag{7b}$$

which suggests that $\Sigma S^{-1}\Sigma S$ is diagonalizable. There exists a nonsingular matrix P such that

$$P^{-1}\Sigma S^{-1}\Sigma S P = \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3, \lambda_4). \tag{8}$$

Since Σ is symmetric and $\Sigma S^{-1}\Sigma S$ is diagonalizable, Σ can be congruent to a diagonal matrix by a symplectic matrix. There exists a transfer matrix R by which the beam with the initially coupled transverse dynamics can be decoupled. In addition, with a similar process, it can also be proved that 6D beam matrices can be decoupled theoretically.

DECOUPLING OF TRANSVERSE-COUPLED DYNAMICS

For the general decoupling of the transverse-coupled dynamics of the beam, an appropriate transfer matrix is required. Usually, four skew quadrupoles are used to decouple the beam because four independent coupling items need to be set to zero [14]. Theoretically, two rotated quadrupoles, which provide 4 variables, can be adopted to decouple the beam.

The transfer matrix of a rotating thin quadrupole lens is

$$R_{\mathrm{rotquad}}(\alpha, f) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \frac{\cos 2\alpha}{f} & 1 & -\frac{\sin 2\alpha}{f} & 0 \\ 0 & 0 & 1 & 0 \\ -\frac{\sin 2\alpha}{f} & 0 & -\frac{\cos 2\alpha}{f} & 1 \end{pmatrix}, \tag{9}$$

where α is the rotation angle of the quadrupole about $+z$-axis, $f = B\rho/(GL_{\mathrm{eff}})$ is the focal length, $B\rho = (m_0 c\beta\gamma)/q$ is the magnetic rigidity of the particle; G is the quadrupole gradient; and L_{eff} is its effective length. The transfer matrix with two rotated quadrupoles,

$$R = R_{\mathrm{rotquad}}(\alpha_2, f_2) \cdot R_{\mathrm{drift}}(L) \cdot R_{\mathrm{rotquad}}(\alpha_1, f_1), \tag{10}$$

where $R_{\mathrm{drift}}(L)$ is the transfer matrix of the drift with length L. The transfer matrix with more rotated quadrupoles can also be given. Thus the angles and focusing strengths of the quadrupoles can be given by solving

$$R\Sigma R^T = \begin{pmatrix} M & 0 \\ 0 & N \end{pmatrix}, \tag{11}$$

where M and N are 2×2 symmetric matrices.

In case that the nonlinear equations have no solutions, more variables should be added into the equations.

EXAMPLE

As is known, a triplet and a skew triplet with six variables of quadrupole gradients can be employed to decouple the beam. We try to find the minimum number of quadrupoles required for decoupling. The transverse emittance growth is adopted to characterize the decoupling capacity:

$$\Delta\varepsilon = \frac{\varepsilon_{1,t} - \varepsilon_{0,t}}{\varepsilon_{0,t}}, \tag{12}$$

where $\varepsilon_{0,t}$ and $\varepsilon_{1,t}$ are the emittance in the transverse plane (x or y) before the coupling and after the decoupling, respectively.

Beamline

The transverse-coupled beam is obtained by one Alvarez-type DTL [15]. The quadrupoles in the DTL are mounted with rotation errors. The rotation errors are uniformly random distributed within ±5°. In the following discussion, a set of fixed errors is used. The beam simulation is performed with TraceWin [16]. The envelope in the coupling DTL shown in Fig. 1. The normalized RMS emittance in x and y plane roses from 0.19π mm·mrad to 1.02π mm·mrad after the DTL. A FODO lattice downstream the DTL is employed for decoupling. The main parameters of the FODO lattice are given in Table 1.

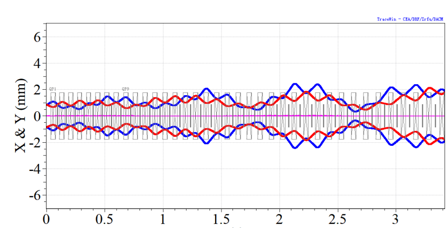

Figure 1: RMS envelope in the coupling DTL.

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Table 1: Main Parameters of FODO Lattice

Parameter	**Value**
Ion type	Proton
Energy	7 MeV
Quadrupole length	0.2 m
Quadrupole gradient	5 T/m
Drift between quads	0.1 m

Two Quads Decoupling

Two rotated quadrupoles are used for decoupling. With fixed rotation angles, the strengths of the quadrupoles are scanned to reach the minimum emittance growth. The minimum emittance growth with different rotation angles is shown in Fig. 2.

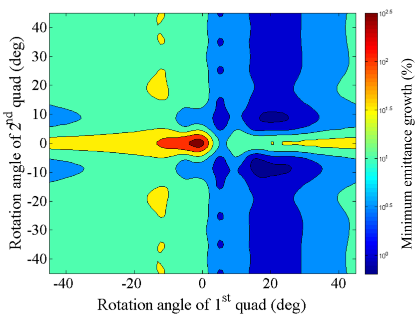

Figure 2: Minimum emittance growth with different rotation angles of decoupling quadrupoles.

Figure 2 shows that the rotation angle of the first decoupling quadrupole needs to be near 15°. And the second should be different from 0° or 90°. Minimum emittance growth is insensitive to the rotation angle of the second if the angle is not near zero. The minimum emittance growth is 1% with two decoupling quadrupoles.

Three or Four Quads Decoupling

The decoupling section can be a 15° quadrupole with a rotated doublet for three quadrupoles decoupling. For four quadrupoles decoupling, the section can be a 15° doublet with a rotated doublet. The emittance growths are 0.8% and 0.1%, for the two cases. The emittance in x or y planes along the decoupling section is shown in Fig. 3.

CONCLUSION

This paper presents evidence of the transfer matrix to decouple the coupled transverse-dynamics of the beam. A minimum number of rotated quadrupoles for decoupling is given. Two rotated quadrupoles can suppress emittance growth to 1% after a coupling section, which is acceptable in general cases.

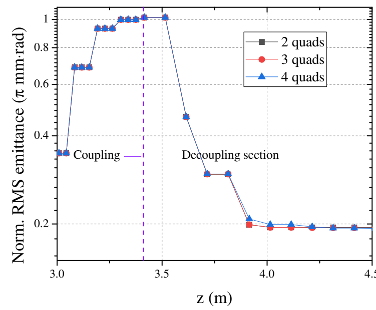

Figure 3: Tranverse emittance along the decoupling section.

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APPENDIX

The inequality $\varepsilon_{4D} \leq \varepsilon_x \varepsilon_y$ can be derived from the volume of the 4D hyperellipsoid. $\pi\varepsilon_{4D}$, $\pi\varepsilon_x$,$\pi\varepsilon_y$ are the volume of 4D emittance hyperellipsoid, the projected ellipse area of the hyperellipsoid onto x and y planes, respectively. The volume of the projected hyperellipsoid with the same principal axes of the projected ellipses on x and y plane is $\pi\varepsilon_x\varepsilon_y$. Two hyperellipsoid are inscribed in the same hypercuboid (Fig. 4). The volume of the projected hyperellipsoid is the largest among the inscribed hyperellipsoids, so

$$\varepsilon_{4D} \leq \varepsilon_x \varepsilon_y. \tag{13}$$

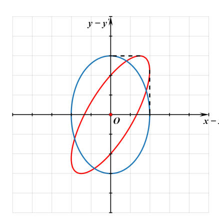

Figure 4: Sketch diagram of the 4D emittance hyperellipsoid (red line) and projected hyperellipsoid (blue line) to illustrate $\varepsilon_{4D} \leq \varepsilon_x \varepsilon_y$

Also, it can be proved that the beam matrix is a positive semidefinite matrix by calculating the leading principal minors of the beam matrix.

$$\Sigma = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{xy}^T & \sigma_{yy} \end{pmatrix}, \tag{14}$$

According to Fischer's inequality [17], for positive semidefinite matrix Σ,

$$\det(\Sigma) \leq \det(\sigma_{xx}) \cdot \det(\sigma_{yy}), \tag{15}$$

and finally $\varepsilon_{4D} \leq \varepsilon_x \cdot \varepsilon_y$ can be deduced.

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