SYMPLECTIC TRACKING FOR THE ROBINSON WIGGLER

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Abstract

A Robinson wiggler (RW) is considered to be installed in the Metrology Light Source (MLS) to lengthen the bunch and improve the Touschek lifetime by manipulating the damping partitions. Symplectic tracking is crucial to study the impact of the nonlinear field components introduced by the Robinson wiggler. This paper introduces a tracking method based on an implicit symplectic integrator to solve the exact Hamiltonian equations of particle motion in the wiggler. In addition, a numerical generating function method is implemented as an approach to realize fast tracking.

INTRODUCTION

The Metrology Light Source (MLS) is an electron storage ring operated at energies from 50 to 630 MeV for metrology applications in the THz to extreme UV spectral range [1]. The Robinson wiggler (RW), a transverse gradient wiggler aiming to control the damping partitions, is considered to be installed at the MLS due to the user's high demands for longer beam lifetime. With the RW in a dispersive straight section, the longitudinal damping can be to transferred to the horizontal plane. As a consequence the bunch can be lengthened and the transverse emittance can be reduced [2]. Using the vertical white noise excitation to keep the transverse beam size unchanged, there is potential to double the lifetime compared to the present value. However, the Robinson wiggler introduces nonlinear distortions to the beam dynamics, which should be studied carefully.

The analysis beam dynamics in the storage ring is based on Hamiltonian mechanics. A specific form for the Hamiltonian in a general set of equations describes the motion for a particular dynamical system. Particle motion at any position in the storage ring can be obtained by solving [3]:

$$H = \frac{\delta}{\beta_0} - \frac{qA_z}{P_0} - \sqrt{\left(\frac{1}{\beta_0} + \delta\right)^2 - \left(p_x - \frac{qA_x}{P_0}\right)^2 - \left(p_y - \frac{qA_y}{P_0}\right)^2 - \frac{1}{\beta_0^2\gamma_0^2}},$$

$$(1)$$

$$\frac{dx_i}{ds} = \frac{\partial H}{\partial p_i}, \qquad (2)$$

$$\frac{dp_i}{ds} = -\frac{\partial H}{\partial x_i}. \qquad (3)$$

where x_i are the coordinates of the particle, p_i are the components of the momentum and H is the Hamiltonian.

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In tracking codes the dipoles and the multipoles are usually modeled with the impulse boundary approximation, in which the magnetic field is assumed to be constant within the effective boundary of the magnet and zero outside. In this model, only the longitudinal component of the vector potential is needed to describe the system. The coordinates and their conjugate canonical momenta are not mixed in the Hamiltonian, so the Hamiltonian could be split into drift-kick combinations [4].

The magnetic field in a wiggler or undulator is three dimensional, in this case the splitting method fails. There is an explicit symplectic integrator developed by Wu, Forest and Robin [5], which requires the Hamiltonian to be expanded in the paraxial approximation. However, the transverse momenta p_x and p_y may reach large values inside the RW due to the low operation energy of MLS, and the paraxial approximation is not longer appropriate. Consequently we use a symplectic Runge-Kutta integrator to solve the exact Hamiltonian equations.

Symplectic Runge-Kutta methods are implicit, and solving algebraic equations at each step inside the wiggler are computationally expensive. Thus a numerical generating function method is implemented to realize fast tracking for nonlinear dynamics studies. The tracking results show that the Robinson wiggler distorts the nonlinear beam dynamics, but it will not be an obstacle to operate the MLS with the RW.

ANALYTICAL REPRESENTATION OF THE MAGNETIC FIELD IN THE ROBINSON WIGGLER

The components of vector potential A_x, A_y, A_z are needed to build the Hamiltionian, therefore the analytical representation of the filed is necessary. The vector potential of a wiggler can be derived from the Halbach expansions of the field expressed in [3, 5]:

$$B_x = -\sum_{m,n}^{M,N} C_{mn} \frac{mk_x}{k_{y,mn}} sin(mk_x x) sinh(k_{y,mn}y) sin(nk_z z),$$

$$(4)$$

$$B_y = \sum_{m,n}^{M,N} C_{mn} cos(mk_x x) cosh(k_{y,mn}y) sin(nk_z z), \quad (5)$$

$$B_z = \sum_{m,n}^{M,N} C_{mn} \frac{nk_z}{k_{y,mn}} cos(mk_x x) sinh(k_{y,mn}y) cos(nk_z z),$$

$$(6)$$

$$k_{y,mn}^2 = m^2 k_x^2 + n^2 k_z^2 \qquad (7)$$

where k_z is the period of the oscillation of the field along the z axis, defined as the reference trajectory in the Cartesian

coordinate system. And k_x determines the transverse "roll off" of the field with increased distance along x axis, while C_{mn} determines the amplitude of the field.

Figure 1 depicts the vertical component of magnetic field in the second period of the RW calculated from RADIA [6]. The B_y values on the median plane are horizontally and longitudinally asymmetric, which are more complicated than expressed in Eq. (5). It is necessary to modify the Halbach expansions by adding angular θ_{mn} and ϕ_{mn} terms to describe the magnetic field of the RW accurately. Then the B_y is expressed in:

$$B_y = \sum_{m,n}^{M,N} C_{mn} cos(mk_x x + \theta_{mn}) cosh(k_{y,mn} y)$$
$$\times sin(nk_z z + \phi_{mn}), \quad (8)$$

accordingly Eq. (4) and Eq. (6) need to be corrected with θ_{mn} and ϕ_{mn} terms to satisfy Maxwell's equations.
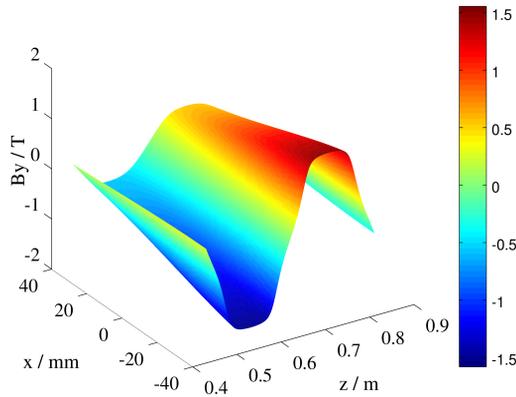
Figure 1: B_y on the median plane (y = 0) in the second period of the RW designed for the MLS.

The coefficients C_{mn}, θ_{mn} and ϕ_{mn} can be obtained from Fourier decomposition of the field component By. As shown in Fig. 2, the residuals of analytical field respresentation with $M = 20$ and $N = 40$ are less than 1‰ for y =10 mm, even below 0.1‰ on the median plane, which fulfill the requirements for the beam dynamics studis.

SYMPLECTIC INTEGRTAOR

The Runge-Kutta method can be used to integrate the Hamiltonian equations of motion; however, the integration will only be symplectic for specific Butcher tableaux [7]. Applying the implicit-midpoint integrator, a second order Runge-Kutta method, Eqs. (2)–(3) can be rewritten as [3]:

$$x(s_n + \Delta s) = x(s_n) + \Delta s \frac{\partial H}{\partial p_x}\Big|_{x=x_n^{(1)}, p_x=p_{xn}^{(1)}}, \quad (9)$$

$$p_x(s_n + \Delta s) = p_x(s_n) - \Delta s \frac{\partial H}{\partial x}\Big|_{x=x_n^{(1)}, p_x=p_{xn}^{(1)}}, \quad (10)$$
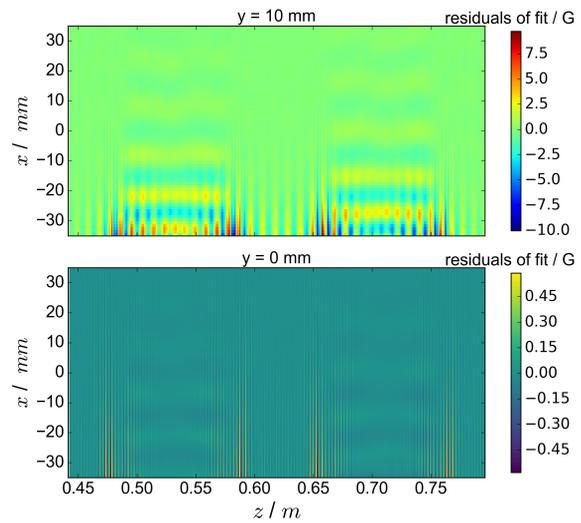
Figure 2: Differences between analytical representation of magnetic field and numerical field map in the second period of the RW.

in which the intermediate values $x_n^{(1)}$ and $p_{xn}^{(1)}$ can be solved from the following:

$$x_n^{(1)} = x(s_n) + \frac{1}{2}\Delta s \frac{\partial H}{\partial p_x}\Big|_{x=x_n^{(1)}, p_x=p_{xn}^{(1)}}, \quad (11)$$

$$p_{xn}^{(1)} = p_x(s_n) - \frac{1}{2}\Delta s \frac{\partial H}{\partial x}\Big|_{x=x_n^{(1)}, p_x=p_{xn}^{(1)}}, \quad (12)$$

with the Newton-Raphson method [8].

The implicit mid-point integrator is applied to track the particles through the second period of the Robinson wiggler. Figure 3 illustrates that the trajectory from sympletic integration is benchmarked by non-symplectic Runge-Kutta method through the numerical field map, which indicates the analytical representation of the field and the symplectic integrator are reliable.
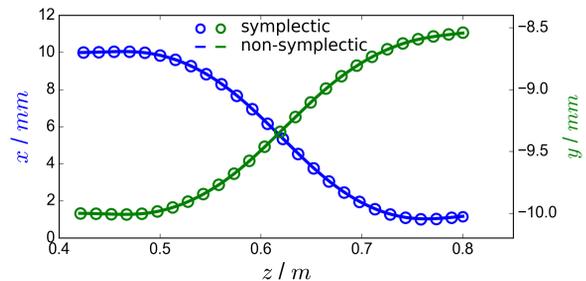
Figure 3: Particle trajectories in the Robinson wiggler from symplectic and non-symplectic integrators.

NUMERICAL GENERATING FUNCTION METHOD

The stepwise implicit integration is very time-consuming, and it is not practical to realize multi-turn particle tracking.

Therefore, a numerical generating function (GF) is introduced to realize fast symplectic tracking. The generating function is built with the initial particle momenta p_{xi}, p_{yi} and the final position variables x_f, y_f, as described in the following [9–11]:.

$$F(q_{xi}, q_{yi}, p_{xf}, p_{yf}) = \sum_{k+l+m+n=1}^{M} a_{klmn} q_{xi}^k q_{yi}^l p_{xf}^m p_{yf}^n,$$ (13)

$$p_{xi} = \frac{\partial F}{\partial q_{xi}}, p_{yi} = \frac{\partial F}{\partial q_{yi}}, q_{xf} = \frac{\partial F}{\partial p_{xf}}, q_{yf} = \frac{\partial F}{\partial p_{yf}}.$$ (14)

The effects of the RW on the electron motion are sampled by tracking a bunch of electrons through the field map at fixed energy. The coefficients a_{klmn} in Eq. (13) are fitted from the initial and final momenta and positions. When the generating function F is built, p_{xf}, p_{yf}, q_{xf}, q_{yf} can be obtained successively by solving the nonlinear equations in Eq. (14). In this paper a 9th-order GF with 714 coefficients is used to model the whole Robinson wiggler. Multi-particle tracking and fitting the coefficients are also computationally expensive, however, these only need to be done once per field map. Moreover, the multi-particle tracking doesn't have to be symplectic due to the intrinsic symplecticity of the generating function.
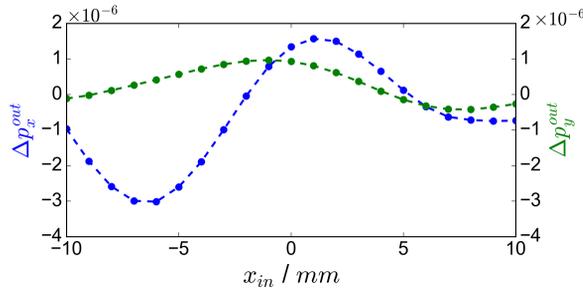
Figure 4: Discrepancy of the horizontal and vertical exit momenta between GF method and Runge-Kutta integration after passing through the whole wiggler at different initial coordinates. The initial y is set to be 2 mm and initial momenta p_x, p_y are 0.

It can be seen from Fig. 4 that the discrepancies of the GF tracking at different initial positions can reach the error level of $\Delta p_x^{out} < 3 \times 10^{-6}$ for a 2.3 m long field map compared to the Runge-Kutta integration method. Although the GF can describe the Robinson wiggler accurately, the accuracy highly depends on the sampled particles. In principle, the p_{xi}, p_{yi}, x_f, y_f should cover a large parameter space.

NONLINEAR DYNAMICS STUDIES

The nonlinear dynamics simulations are realized with the ELEGANT code [12]. Although there is no module in elegant which can represent the Robinson wiggler directly, its SCRIPT element provides an interface to use customized integrator for an elaborate device like the RW and make use of the powerful analysis tools in ELEGANT. The frequency

map analysis of the Robinson wiggler on the nonlinear beam dynamics is shown in Fig. 5. As a comparison, the dynamic aperture (DA) in standard user mode at the MLS is larger than the transverse size of the octagonal vacuumm chamber with 42 mm in hight and 70 mm in width [2]. With the RW turned on, the DA is shrinks significantly, but it is sufficient for the operation.
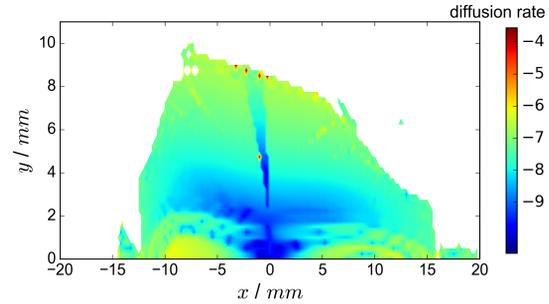
Figure 5: On-momentum dynamic aperture of the MLS storage ring with the RW

SUMMARY

The analytical representation of magnetic field in the RW has been established based on Halbach expansions, which can achieve 10^{-3} relative field accuracy. Based on this, the exact Hamiltonian can be built and equations of particle motion can solved by a second-order symplectic Runge-Kutta integrator. The integration needs to solve the nonlinear equations implicitly step by step, therefore it is very time-consuming and not practical in multi-turn tracking. As an alternative approach, the GF method shows the advantages of speed and symplecticity. However, the coefficients of GF must be fitted carefully and the accuracy should be benchmarked with other symplectic integrators. The nonlinear dynamics study is performed with the ELEGANT code, which allows to call an external code using the GF method for the RW. The tracking results show that the RW has nonnegligible impact on the dynamic aperture, however the dynamic aperture is sufficient for operating the RW at the MLS.

ACKNOWLEDGEMENT

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