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ELECTROMAGNETIC IMPULSE OF BEAM DENSITY F(X,Y)G(Z)*

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Abstract

We calculate the transverse impulse on a test particle as a bunch of charged particles beam passes by. It is often assumed, but seldom proven, that the EM field from a beam density distribution factored into transverse and longitudinal parts, F and G respectively, has also a factored form P(x,y)Q(z). This factorization is not possible for stationary charges. Contrastingly, it becomes increasingly accurate for ultra-relativistic particle beams. We give a general analysis, show how to develop the corrections in terms of integrals of F and derivatives of G. What is significant is that if we integrate over longitudinal coordinate z to find the transverse impulse on a witness charge, the correction terms integrate to zero leading to the impulse P(x,y)Integral[Q(z)] independent of bunch shape. If this result is already known, this paper serves as a reminder.

INTRODUCTION

The space-charge (SC) tune shift and beam-beam (BB) tune shift share common origins, but significant differences. For the SC shift, test particle travels with the beam resulting in (i) partial cancellation of the electric and magnetic forces; and (ii) effect is continuous rather than impulsive. For the BB shift, test particle moves in the opposite direction to beam resulting in:(i) addition of electric and magnetic forces; and (ii) effect is impulsive whenever two bunches pass one another.

We can point to two regimes for this type of calculation: (i) the long bunch regime as typified by the CERN Intersecting Storage Rings [1]; and (ii) the short bunch regime with longitudinal and transverse dimensions comparable. The CERN Large Hadron Collider (LHC) starts to approach the latter inside the interaction regions (but not at the interaction point). There the longitudinal r.m.s. size is $\sigma_z \approx 8$ cm, lattice β is around 500 m and normalized emittance ≈4 μm leading to transverse r.m.s. size $\sigma_r \approx 0.05$ cm.

For the model of the incoherent beam-beam interaction (a.k.a. weak-strong model), we need the transverse impulse imparted to the test particle (in the weak beam) located at (x,y,z=0) as a strong-beam bunch passes by. Hence, we need the transverse electromagnetic field $EM^{\perp}$(x,y,z, time) from a 3D bunch (x,y,z+v.t), and to integrate this over time.

This paper is concerned with the regime $\sigma_z \sim \sigma_r$. In the case that the density factors as F(x,y)G(z) it transpires that the impulse if very similar to the long-bunch case, and increasingly so at high kinematic γ.

Let **v** and **u** be the velocity of the source and observer, respectively. The bunch centre is located at (z + v.t)=0, so we can replace integral ∫dt with ∫dz/v.

We break the calculation into two parts: (1) the contributions when the bunch is downstream and upstream of the test particle; and (2) when the witness particle is inside the bunch. The two parts must be added together.

FAR-FIELD IMPULSE

There are fields (outside the charge distribution) ahead of and behind the bunch. Moving charges produce electric **E** and magnetic **B** fields. The force vector is **F**= **E**+**u**×**B**, where **u** is velocity of observer in laboratory frame.

For a bunch moving leftward and counter-moving (or stationary) test charge, the forces contribute an impulse.

$$\int_{z=\mathrm{tail}}^{z=-\infty} F\,dz + \int_{z=\infty}^{z=\mathrm{head}} F\,dz$$

When **u**=**v**, test charge and beam co-moving, the range of integration is invalid; there is no impulse.

To do these integrals, we replace the bunch by concentrated charge at its centroid, and use the expressions from Jackson [2] for transformation of the fields of relativistic moving charges. Let observer be at distance "x" transverse to the trajectory of the moving charges. Let total bunch charge be Q and 4πε≡ 1. Let bunch extend from head zh to tail zt (moduli). In cylinder coordinates, the fields are

$$E_r = \frac{Qx\gamma}{(x^2+z^2\gamma^2)^{3/2}}$$
$$E_z = \frac{Qz\gamma}{(x^2+z^2\gamma^2)^{3/2}}$$
$$B_\phi = -\frac{vE_r}{c^2}$$

Longitudinal Impulse

The force is due to E_z alone ($F_z = E_z$). Performing the integral we find:

$$\int F_z\,dz = -\frac{Q}{\gamma\sqrt{x^2+\gamma^2 z_h^2}} + \frac{Q}{\gamma\sqrt{x^2+\gamma^2 z_t^2}}$$
$$\int F_z\,dz \approx \frac{Q(z_h - z_t)}{\gamma^2 z_h z_t} - \frac{Qx^2(z_h^3 - z_t^3)}{2\gamma^4 z_h^3 z_t^3}$$

The impulse tends to zero when γ>>1, and is zero when $z_h = z_t$. This is consistent with the expectation that the entire impulse is identically zero, irrespective of γ >1.

Transverse Impulse

The force is due to E_r and $u\times B_\varphi$. When *u*= -*v*, test charge and beam oppositely directed, the electric and magnetic forces add up. The net result is $E_r(1+uv/c^2) \to 2E_r$. Hence the force is $F_r = 2E_r$ in the relativistic regime. Performing the integral we find:

$$\int F_r\,dz = \frac{2Q}{x}\left(2 - \frac{\gamma z_h}{\sqrt{x^2+\gamma^2 z_h^2}} - \frac{\gamma z_t}{\sqrt{x^2+\gamma^2 z_t^2}}\right)$$

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$$\int F_r\, dz \approx \frac{Qx}{\gamma^2}\left(\frac{1}{z_h^2}+\frac{1}{z_t^2}\right)$$

Here x is less than the bunch transverse extent b ($0<x<b$). The impulse tends to zero when the bunch is much longer than its width (z_h, z_t >>b) and when γ>>1. Hence the contribution is insignificant in both the long-bunch and short - bunch regimes. Note the $1/\gamma^2$ dependence is not from EM cancellation, rather it follows from the relativistic reduction of fields in front and behind the bunch.

When **u**=**v**, test charge and beam co-moving, the range of integration is invalid.

NEAR-FIELD IMPULSE

We come now to the condition where the test particle is within the bunch. As we shall see, the form of the EM field only separates into a product when the charges are moving.

Field from Stationary Charge Distribution

Suppose the charge density is separated: R=λ(z)ρ(x,y). The total charge is Q =∫R(x,y,z)dVol = ∫λ(z)dz∫ρ(x,y)dS. The number of particles is N=Q/q, where q is the individual charge.

For stationary charges, Div.**E** = R(x,y,z)/ε and Curl **E**=0 and **B**=**0**. These equations are sufficiently constraining that it is not possible to find solutions (inside the bunch) of the product form: $E^{\perp}$ = λ(z) h(x,y) and $E_{\parallel}$ = (∂λ/∂z) g(x,y).

Fields from Moving Charge Distribution

We calculate the fields in the laboratory frame for a moving charge distribution, starting from Maxwell's equations. The charge density and current sources are, respectively: R= ρ(x,y)λ(z + vt) and **J** = $\mathbf{e_z}$ (-v) R(x,y,z,vt).

$$\nabla\cdot\mathbf{E}=\frac{\rho}{\varepsilon_0}$$

$$\nabla\cdot\mathbf{B}=0$$

$$\nabla\times\mathbf{E}=-\frac{\partial\mathbf{B}}{\partial t}$$

$$\nabla\times\mathbf{B}=\mu_0\mathbf{J}+\mu_0\varepsilon_0\frac{\partial\mathbf{E}}{\partial t}$$

Figure 1: Maxwell's equations for electromagnetism.

The RHS of the two curl equations generates additional geometric freedom that allows a product form for fields. We assume the source (the "strong" beam) moves to the left with velocity -v. The partial derivative ∂/∂t = +v ∂/∂z for bunch moving to left, assuming coordinate z points to right.

Substituting the factored **R** and **J**, we find the condition:

Curl **B** = (v/c²)[-λ(z)$\mathbf{e}_z$ (Div$^{\perp}$.$\mathbf{E}^{\perp}$) + ∂λ(z,t)/∂z $\mathbf{E}^{\perp}$(x,y)]

There is a fully self-consistent solution of the four Maxwell equations, Fig.1, of the form:

$$\mathbf{E}= \mathbf{e_r}\, E_r[r,z] + \mathbf{e_z}\, E_z[r,z]$$

$$\mathbf{B}(\perp)= (-v/c^2)\, \mathbf{e_z} \times\mathbf{E}(\perp)$$

$$E_z = \frac{1}{\gamma^2}\int_0^r (\partial E_r/\partial z)dr$$

$$\mathbf{B}= \mathbf{e}_\varphi\, B_\varphi\,(r,z)$$

$$B_\varphi = (-v/c^2)\, E_r.$$

All quantities follow from the transverse electric field Er, which must satisfy the equation:

$$\frac{1}{r}\mathrm{Er}[r,z]+\frac{1}{\gamma^2}\int \partial^2\mathrm{Er}/\partial z^2\,\mathrm{d}r+\frac{\partial\mathrm{Er}}{\partial r}=\lambda[z]\rho[r]$$

Note, the $1/\gamma^2$ dependence here does not arise from cancellation between electric field E and magnetic field B. Rather, it arises from the field lines of a single charge flattening into a pancake shape, as per the formulae reported by Jackson.

Transverse Impulse

Ultra-relativistic regime

When γ >>1, Er has solution in the desired product form:

$$\mathrm{Er}[r,z]\rightarrow\lambda[z]\ \mathrm{Er}[\mathrm{r}]$$

$$\mathrm{Er}[r]\rightarrow\frac{1}{r}\int_0^r \mathrm{x}\,\rho[x]\,\mathrm{dx}$$

The force acting is F[r,z] =(E_r ±uB_φ) =λ[z]E_r[r](1±uv/c²), with the sign ±depending on the test particle be counter-rotating or co-moving. The impulse is obtained by integrating the bunch over longitude z. The integral is independent of the bunch shape, because it is simply the bunch total charge.

Non-relativistic regime

We may approximate the regime γ>1, by substituting the known form Er[r,z] immediately above into the integro-differential equation and treating it as a source term. Hence to higher order (better than 2nd order), we have

$$\frac{1}{r}\mathrm{Er}[r,z]+\frac{1}{\gamma^2}\int\lambda''[z]\frac{\int_0^r \mathrm{x}\,\rho[x]\,\mathrm{dx}}{r}\mathrm{d}r+\frac{\partial\mathrm{Er}}{\partial r}$$
$$=\lambda[z]\,\rho[r]$$

We now integrate this equation over the longitudinal coordinate from head to tail of the bunch. The integral of $\lambda[z]$ is related to the number of particles in the bunch. Contrarily the integral over the bunch of any derivative of $\lambda[z]$ must be zero, because the bunch goes to zero at its ends.

Hence the electric field contribution to the transverse impulse satisfies:

$$\frac{1}{r}\mathrm{Er}[r]+\frac{\partial\mathrm{Er}}{\partial r}=\frac{N\rho[r]}{\int_0^b r\rho[r]\,dr}$$

$$\mathrm{Er}[r]=N\frac{\int_0^r x\rho[x]\,dx}{r}$$

And like wise for B_φ[r]= ∫B_φ[r,z]dz integrated over z.

Hence the impulse ∫Fdz = E[r] (1+v²/c²) →2 E[r] when counter-moving, and ∫Fdz = E[r] /γ² when co-moving.

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The dependence on u =±v is from magnetic enhancement or cancellation.

SCALING LAW

We have been Cavalier; we assumed that the relativistic kinematic γ is much larger than the derivative of the bunch shape. We revisit this assumption with more care. We take new dimensionless variables that are scaled by the bunch r.m.s transverse and longitudinal dimensions *a* and *b*, respectively. Let *x=r/a* and *y=z/b*. We transform the derivatives according to the chain rule. And then we perform the Fourier transform w.r.t. the dimensionless wavenumber *k*, that is to say we multiply by Exp[*i k y*] and integrate over *y*. The result is that spatial density λ[z] is replaced by spectral density λ[k].

$$\frac{\mathrm{Ex}[x,k]}{x} - q^2 \int_0^x \mathrm{Ex}[u,k]\,\mathrm{d}u + \frac{\partial \mathrm{Ex}[x,k]}{\partial x} = a\,\lambda[k]\rho[x]$$

Here $q^2 \to \frac{a^2k^2}{b^2\gamma^2}$ *is the relevant scale parameter.*

This is an inhomogeneous Bessel equation, and solution may be obtained in terms of Green's functions constructed from Bessel functions; see the Appendix. Inverting the Fourier transform and coordinate scalings leads to the fields as before. Let *e*[*iky*] denote the exponential. If the bunches are periodic, the result is a Fourier series.

$$\mathrm{E}[r,y] = \sum_{k=0}^{\infty} e[iky]\mathrm{Er}[r,k]\lambda[k]$$

If this is integrated over longitude *y* to obtain the impulse, it is evident that only the term in *k*=0 will remain. Hence the contribution of the electric field to the impulse is

$$\mathrm{E}[r] = \int \mathrm{E}[r,y]\,dy = \mathrm{Er}[r,k=0]\lambda[k=0]$$

Now the spectral density λ[k=0] =∫ λ[z]dz is simply the bunch charge. Further, when scale parameter *q*=0, the Bessel equation reduces to simple form with solution

$$\mathrm{Ex}[x,k=0] \to \frac{1}{x}\int_0^x \mathrm{u}\,\rho[u]\,\mathrm{d}u$$

Hence the electric impulse is again

$$\mathrm{Er}[r] = N\frac{\int_0^r x\rho[x]\,dx}{r}$$

And likewise, the magnetic contribution is $\pm uv/c^2\mathrm{Er}[r]$.

Iterative Solution

Previously we had proposed substituting the solution when q=0 into the integro-differential equation and treating it as a source term. Essentially this is an iterative solution. We now do this explicitly for a few simple cases, making use of the dimensionless variables. For brevity, let Fx_0 be equal to Ex/$\lambda[k]$ @ q=0.

ρ(u)	Fx_0	Ex/$\lambda[k]$ @ q≠0
1	ax	$\frac{1}{8}aq^2x^3 + \mathrm{Fx}_0$
1-u	$ax(3-2x)$	$aq^2x^3\left(\frac{3}{8}-\frac{2x}{15}\right) + \mathrm{Fx}_0$
1-u^2	$ax(2-x^2)$	$aq^2x^3\left(\frac{1}{4}-\frac{x^2}{24}\right) + \mathrm{Fx}_0$

The range of *u* & *x* =[0,1].

CONCLUSION

This paper investigates the transverse electro-magnetic impulse due to the passage of a bunch of charged particles under the assumption the charge density is the product form R=λ(z)ρ(x,y). In the long-bunch regime, the electric fields are rigorously of the form λ(z)P(x,y); and the impulse is independent of the bunch shape. In the short-bunch regime, the electric field is not strictly of the product form but may be written as a Fourier series. In first approximation, the coefficients in this series may be found as an expansion in $q = \frac{ak}{b\gamma} < 1$.

APPENDIX

Let s = ± i q where i = √-1. The Green's function with boundary condition ∂Ex[x,k]/∂x = a ρ[0]/2 at x=0 is:

$$\mathrm{Ex}[x,k] \to \left(\frac{a}{s}\right)\mathrm{BesselJ}[1,sx]\rho[0] + \frac{1}{2}a\pi\int_0^x u(-\mathrm{BesselJ}[1,sx]\mathrm{BesselY}[1,su] + \mathrm{BesselJ}[1,su]\mathrm{BesselY}[1,sx])\rho'[u]\,du$$

REFERENCES

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