ISBN: 978-3-95450-184-7 doi:10.18429/JACoW-IPAC2018-THPMK054

ANALYSIS OF 1D FEL SIDEBAND INSTABILITY WITH INCLUSION OF ENERGY DETUNE AND SPACE CHARGE*

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Abstract

It has been known that free-electron laser (FEL) is capable of generating a coherent high-power radiation over a broad spectrum. Recently there is a great interest in pursuing higher peak power (for example, at terawatt level) in FEL that can enable coherent diffraction imaging and probe fundamental high-field physics. The FEL radiation power can be increased by virtue of undulator tapering. However the FEL sideband signal begins to exponentially grow in the post-saturation regime. In this paper we extend our sideband analysis [1] by including both the initial energy detune and longitudinal space charge in an effective 1-D model. A dispersion relation with explicit space charge parameter is derived. To first order, the effect of energy detune does not participate in the resultant dispersion relation. The study is carried out semi-analytically and compared with numerical solutions.

INTRODUCTION

In this paper as a first step we formulate the one-dimensional (1-D) high-gain free-electron laser (FEL) system with inclusion of energy detuning following Bonifacio *et al.* [2–4]. Then we incorporate the effect of the longitudinal space charge in the sideband instability analysis. In particular, we treat the case for high-gain FEL with undulator tapering. By using a single macroparticle perturbation analysis, we derive a quartic dispersion relation that is useful in analyzing sideband instabilities. Throughout the analysis we use Gaussian units unless stated otherwise.

FEL EQUATIONS WITH DETUNING AND LONGITUDINAL SPACE CHARGE

To include the effect of energy detuning, it is convenient to introduce the following variables with scaling [2]:

$$\tilde{\theta}_j = \theta_j - \theta_0' z = \theta_j - \theta_0' \frac{\bar{z}}{2k_u\rho} \tag{1}$$

$$\mathcal{E} = Ae^{i\delta\bar{z}} = \frac{\omega}{\omega_p\sqrt{\rho\gamma_R(0)}} a e^{i\delta\bar{z}} \tag{2}$$

where $\theta_j = (k+k_u)z_j - \omega t_j$ is the ponderomotive phase, i.e., the phase of j-th electron relative to the resonant radiation field ($\omega = kc$ and $k_u = 2\pi/\lambda_u$ with λ_u the undulator period), $a = \frac{eE}{mc^2k}$ is the scaled (complex) radiation field, $\theta_0' = \frac{d\theta}{dz}\big|_{z=0} = k_u\left(1-\frac{\gamma_R^2(0)}{\gamma_0^2}\right)$, $\gamma_0 = \langle\gamma\rangle_0 = \frac{1}{N_e}\sum_{j=1}^{N_e}\gamma_j(0)$, γ_R is the resonant electron energy, N_e the total number of electrons, $A = \frac{\omega}{\omega_p\sqrt{\rho\gamma_R(0)}}a$, $\bar{z} = 2k_u\rho z$, $\bar{t} = 2k_u\rho t$, where $\rho = \frac{1}{\gamma_R(0)}\left(\frac{K(0)}{4\sqrt{2}}\frac{\omega_p}{ck_u}\right)^{2/3}$ is the FEL or Pierce parameter with $\omega_p = \sqrt{\frac{4\pi n_e e^2}{m}}$ the nonrelativistic plasma frequency and $K(0)$ the dimensionless undulator parameter. Then the shifted ponderomotive phase takes the form

$$\tilde{\theta}_j = \theta_j - \frac{1}{2\rho}\frac{\gamma_0^2-\gamma_R^2(0)}{\gamma_0^2}\bar{z} \approx \theta_j - \frac{1}{2\rho}\frac{\gamma_0^2-\gamma_R^2(0)}{\gamma_R^2(0)}\bar{z} \equiv \theta_j - \delta\bar{z}$$

Here we introduce the detuning parameter as $\delta = \frac{1}{2\rho}\frac{\gamma_0^2-\gamma_R^2(0)}{\gamma_R^2(0)} \approx \frac{1}{\rho}\frac{\gamma_0-\gamma_R(0)}{\gamma_R(0)}$ which signifies an offset of the average energy of electrons from the resonant energy at the entrance of an undulator. It has been known that the longitudinal space charge field will result in the energy modulation [3]

$$\left(\frac{d\gamma_j}{dz}\right)_{LSC} = -ik\left(\frac{\omega_p}{\omega}\right)^2\left\langle e^{-i\theta}\right\rangle e^{i\theta_j} + c.c. \tag{3}$$

With these new scaled variables, we find [2]

$$\begin{aligned}\left(\frac{\partial}{\partial\bar{z}}+\frac{1}{\bar{v}_z}\frac{\partial}{\partial\bar{t}}\right)\tilde{\theta}_j &= \frac{1}{2\rho}\left(1-\frac{1}{\rho^2\Gamma_j^2}\frac{\gamma_R^2(z)}{\gamma_R^2(0)}\right)\\ &+ i\frac{1}{\rho}\left(\mathcal{E}\frac{e^{i\tilde{\theta}_j}}{\Gamma_j^2}-\text{c.c.}\right)-\frac{\sigma}{2}\frac{|\mathcal{E}|^2}{\Gamma_j^2},\end{aligned} \tag{4}$$

$$\begin{aligned}\left(\frac{\partial}{\partial\bar{z}}+\frac{1}{\bar{v}_z}\frac{\partial}{\partial\bar{t}}\right)\Gamma_j &= -\frac{1}{\rho}\left(\mathcal{E}\frac{e^{i\tilde{\theta}_j}}{\Gamma_j}+\text{c.c.}\right)\\ &- i\sigma\left(\langle e^{-i\tilde{\theta}}\rangle e^{i\tilde{\theta}_j}-\text{c.c.}\right),\end{aligned} \tag{5}$$

and

$$\left(\frac{\partial}{\partial\bar{z}}+\frac{1}{c}\frac{\partial}{\partial\bar{t}}\right)\mathcal{E} = \frac{1}{\rho}\left\langle\frac{e^{-i\tilde{\theta}}}{\Gamma}\right\rangle + i\left(\delta-\frac{\sigma}{2}\left\langle\frac{1}{\Gamma}\right\rangle\right)\mathcal{E}. \tag{6}$$

Here $\Gamma_j(z) = \frac{\gamma_j(z)}{\rho\gamma_R(0)}$ and $\sigma = 4\rho\frac{2+K^2(0)}{K^2(0)}$

* The work was supported by the US Department of Energy (DOE) under contract DE-AC02-76SF00515 and the US DOE Office of Science Early Career Research Program grant FWP-2013-SLAC-100164.
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ISBN: 978-3-95450-184-7 doi:10.18429/JACoW-IPAC2018-THPMK054

We now make the coordinate transformations

$$\bar{z}' = \bar{z}, \qquad \bar{t}' = \bar{t} - \frac{\bar{z}}{\bar{v}_z} \tag{7}$$

Having further replaced Γ with η by $\eta(\bar{z}) = \frac{\gamma(\bar{z})-\gamma_R(0)}{\gamma_R(0)} = \rho\Gamma(\bar{z}) - 1$ and linearizing in η, the resulting expressions become

$$\frac{\partial\tilde{\theta}_j}{\partial\bar{z}'} \approx \frac{\eta_j}{\rho} + i\rho(\mathcal{E}e^{i\tilde{\theta}_j} - \mathcal{E}^* e^{-i\tilde{\theta}_j}) - \frac{\sigma}{2}|\mathcal{E}|^2\rho^2(1-2\eta_j), \tag{8}$$

$$\begin{aligned}\frac{\partial\eta_j}{\partial\bar{z}'} &\approx -\rho(1-\eta_j)(\mathcal{E}e^{i\tilde{\theta}_j} + \mathcal{E}^* e^{-i\tilde{\theta}_j}) \\ &- i\rho\sigma\left(\langle e^{-i\tilde{\theta}}\rangle e^{i\tilde{\theta}_j} - \text{c.c.}\right),\end{aligned} \tag{9}$$

and

$$\begin{aligned}&\left(\frac{\partial}{\partial\bar{z}'} - \frac{1}{c\bar{\beta}_z}(1-\bar{\beta}_z)\frac{\partial}{\partial\bar{t}'}\right)\mathcal{E} \\ \approx\ & i\delta\mathcal{E} + \langle e^{-i\tilde{\theta}}(1-\eta_j)\rangle - i\frac{\sigma\rho}{2}\langle 1+\eta_j\rangle\mathcal{E}.\end{aligned} \tag{10}$$

The above formulation assumes a constant undulator K. In the next section we will extend the formulation to the case with undulator tapering $K(z)$.

FEL EQUATIONS FOR TAPERED UNDULATORS

To formulate the FEL equations in the presence of undulator tapering, we define the variables $f_B(z) = \frac{B_u(z)}{B_u(0)} = \frac{K(z)}{K(0)}$ and $f_R(z) = \frac{\gamma_R(z)}{\gamma_R(0)}$ with the resonance condition explicitly expressed as $f_R(z) = \sqrt{(1+K^2(z)/2)/(1+K_0^2/2)}$. Here $K_0 = K(0)$.

Let us write γ_j to first order, i.e., $\gamma_j(\bar{z}) = \gamma_R(\bar{z}) + \Delta\gamma_j(\bar{z})$ with $\Delta\gamma_j \ll \gamma_R$, we find $\frac{1}{2\rho}\left(1 - \frac{1}{\rho^2\Gamma_j^2}f_R^2(\bar{z})\right) \approx \frac{1}{f_R(\bar{z})}\left[\hat{\eta}_j(\bar{z}) - \hat{\eta}_R(\bar{z})\right]$, with definition of $\hat{\eta}(\bar{z}) = \frac{\eta(\bar{z})}{\rho} = \frac{\gamma(\bar{z})-\gamma_R(0)}{\rho\gamma_R(0)} = \Gamma(\bar{z}) - \frac{1}{\rho}$. We then obtain the tapered FEL equations

$$\begin{aligned}\frac{\partial\tilde{\theta}_j}{\partial\bar{z}} &= \frac{1}{f_R(\bar{z})}\left[\hat{\eta}_j(\bar{z}) - \hat{\eta}_R(\bar{z})\right] \\ &+ i\rho\frac{f_B(\bar{z})}{f_R(\bar{z})}\left(\mathcal{E}e^{i\tilde{\theta}_j} - \mathcal{E}^* e^{-i\tilde{\theta}_j}\right) - \frac{\sigma|\mathcal{E}|^2\rho^2}{2f_R^2(\bar{z})},\end{aligned} \tag{11}$$

$$\frac{\partial\hat{\eta}_j}{\partial\bar{z}} = -\frac{f_B(\bar{z})}{f_R(\bar{z})}\left(\mathcal{E}e^{i\tilde{\theta}_j} + \mathcal{E}^* e^{-i\tilde{\theta}_j}\right) - i\sigma\left(\langle e^{-i\tilde{\theta}}\rangle e^{i\tilde{\theta}_j} - \text{c.c.}\right), \tag{12}$$

and

$$\left(\frac{\partial}{\partial\bar{z}} + \frac{\partial}{\partial\bar{u}}\right)\mathcal{E} = \frac{f_B(\bar{z})}{f_R(\bar{z})}\left\langle e^{-i\tilde{\theta}}\right\rangle + i\mathcal{E}\delta - \frac{i\sigma\rho}{2f_R(\bar{z})}\mathcal{E}. \tag{13}$$

In the above equation we have introduced the variable $\bar{u} = \frac{\bar{z}-c\bar{\beta}_z\bar{t}}{1-\bar{\beta}_z}$. Equations (11) to (13) are the 1-D tapered FEL system equations. In what follows we will apply the single macroparticle perturbation to derive the sideband dispersion relation.

SIDEBAND DISPERSION RELATION

For stability analysis we will neglect the second term on the right-hand side of Eq. (11) because it is the effect of radiation on the electron trajectory which is small. For the electrons near resonance, we let

$$\begin{aligned}\mathcal{E}(\bar{z}) &= \left[\mathcal{E}_0(\bar{z}) + \alpha(\bar{u},\bar{z})\right]e^{i\phi(\bar{z})} \\ &= \left[\mathcal{E}_0(\bar{z}) + \alpha'(\bar{u},\bar{z}) + i\alpha''(\bar{u},\bar{z})\right]e^{i\phi(\bar{z})},\end{aligned} \tag{14}$$

$$\hat{\eta}(\bar{z}) = \hat{\eta}_R(\bar{z}) + \delta\hat{\eta}(\bar{u},\bar{z}), \tag{15}$$

$$\tilde{\theta}(\bar{z}) = \tilde{\theta}_R(\bar{z}) + \delta\tilde{\theta}(\bar{u},\bar{z}). \tag{16}$$

Assuming $\tilde{\theta}_R + \phi = \psi_R$ is constant which corresponds to quadratic tapering [5,7], we have $\frac{\partial\tilde{\theta}_R}{\partial\bar{z}} = -\frac{\partial\phi}{\partial\bar{z}}$. Then we get the phase equation to first order in $\delta\hat{\eta}$, $\delta\tilde{\theta}$, α', and α'' as

$$\frac{\partial\delta\tilde{\theta}}{\partial\bar{z}} \approx \frac{1}{f_R}\delta\hat{\eta} - \frac{\sigma\rho^2}{2f_R^2}\left(\mathcal{E}_0^2 + 2\mathcal{E}_0\alpha'\right) + \frac{\partial\phi}{\partial\bar{z}} \tag{17}$$

Similarly for the energy equation we obtain

$$\frac{\partial\langle\delta\hat{\eta}\rangle}{\partial\bar{z}} = 2\frac{f_B}{f_R}\left(\mathcal{E}_0\langle\delta\tilde{\theta}\rangle\sin\psi_R - \alpha'\cos\psi_R + \alpha''\sin\psi_R\right), \tag{18}$$

where $\langle e^{i\delta\tilde{\theta}}\rangle \approx (1 + i\langle\delta\tilde{\theta}\rangle)$ has been used.

Let us now consider the wave equation. Inserting Eqs. (14) and (16) into Eq. (13) and extracting only the linear terms, we have

$$\begin{aligned}&\left(\frac{\partial}{\partial\bar{z}} + \frac{\partial}{\partial\bar{u}}\right)\alpha' + i\left(\frac{\partial}{\partial\bar{z}} + \frac{\partial}{\partial\bar{u}}\right)\alpha'' - \alpha''\frac{\partial\phi}{\partial\bar{z}} + i\alpha'\frac{\partial\phi}{\partial\bar{z}} \\ =\ & \frac{f_B}{f_R}(-\langle\delta\tilde{\theta}\rangle\sin\psi_R - i\langle\delta\tilde{\theta}\rangle\cos\psi_R) \\ +\ & i\alpha'\left(\delta - \frac{\sigma\rho}{2f_R}\right) - \alpha''\left(\delta - \frac{\sigma\rho}{2f_R}\right).\end{aligned} \tag{19}$$

Note that to zeroth order we have $\frac{\partial\phi}{\partial\bar{z}} = -\frac{1}{\mathcal{E}_0}\frac{f_B}{f_R}\sin\psi_R + \delta - \frac{\sigma\rho}{2f_R}$. Enumerating real and imaginary parts of Eq. (19) separately and taking ensemble averages over particles, we obtain

$$\left(\frac{\partial}{\partial\bar{z}} + \frac{\partial}{\partial\bar{u}}\right)\alpha' + \frac{1}{\mathcal{E}_0}\frac{f_B}{f_R}\alpha''\sin\psi_R = -\frac{f_B}{f_R}\langle\delta\tilde{\theta}\rangle\sin\psi_R, \tag{20}$$

$$\left(\frac{\partial}{\partial\bar{z}} + \frac{\partial}{\partial\bar{u}}\right)\alpha'' - \frac{1}{\mathcal{E}_0}\frac{f_B}{f_R}\alpha'\sin\psi_R = -\frac{f_B}{f_R}\langle\delta\tilde{\theta}\rangle\cos\psi_R. \tag{21}$$

Now we have a set of four linearized equations, (17), (18), (20), and (21) and these can be used to analyze FEL sideband effect. For this purpose, we search for solutions of the form

$$\left(\langle\delta\tilde{\theta}\rangle,\ \langle\delta\hat{\eta}\rangle,\ \alpha',\ \alpha''\right) = \text{Re}\left[g_j(\bar{z})\exp\left\{i\left(\int_0^{\bar{z}} k(\bar{z}')d\bar{z}' - \kappa\bar{u}\right)\right\}\right]$$

where $j = 1,2,3,4$ and we assume k is complex and κ is real and g_j's are functions of $\bar{z}$ but they are slowly varying over the characteristic length scale which is the synchrotron

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Table 1: Relevant Beam and Undulator Parameters for the Sideband Analysis

Name	**LCLS 4.5 keV**	**Ref.** [6]	**Unit**
Peak current	4000	5000	A
Electron beam energy	10064	3.06	MeV
Normalized emittance	0.3	1	μm
Relative energy spread	10^{-4}	0.05	
Average beam size	8.7	300	μm
Undulator K	3.5	0.8	
Undulator period	3	2	cm
Resonant wavelength	0.0002755	338	μm
Pierce parameter	1.57×10^{-3}	0.174	

oscillation period, so we can neglect terms like $\frac{\partial g_i}{\partial \bar{z}}$ and $\frac{\partial k}{\partial \bar{z}}$ and seek for the zeroth-order solution. As a result, we find

$$ikg_1 - \frac{1}{f_R} g_2 + \frac{\sigma\rho^2}{f_R^2} \mathcal{E}_0 g_3 = 0, \tag{22}$$

$$2\frac{f_B}{f_R} g_1 \mathcal{E}_0 \sin\psi_R - ikg_2 - 2\frac{f_B}{f_R} g_3 \mathcal{E}_0 \cos\psi_R + 2\frac{f_B}{f_R} g_4 \sin\psi_R = 0, \tag{23}$$

$$\frac{f_B}{f_R} g_1 \sin\psi_R + i(k-\kappa) g_3 + \frac{1}{\mathcal{E}_0}\frac{f_B}{f_R} g_4 \sin\psi_R = 0, \tag{24}$$

and

$$\frac{f_B}{f_R} g_1 \cos\psi_R - \frac{1}{\mathcal{E}_0}\frac{f_B}{f_R} g_3 + i(k-\kappa) g_4 = 0. \tag{25}$$

These are homogeneous algebraic equations for g_i and a condition for nontrivial solutions to exist requires that the determinant must be zero. Therefore we have the FEL sideband dispersion equation

$$(k - \Omega_{\text{syn}}^2)\left[(k-\kappa)^2 - \frac{f_R^2}{4\mathcal{E}_0^4}\Omega_{\text{syn}}^4\right] - \frac{f_B^2}{\mathcal{E}_0^2 f_R^2}\Omega_{\text{syn}}^2$$
$$-i\frac{k\sigma\rho^2}{\mathcal{E}_0}\frac{f_B}{f_R^2}\Omega_{\text{syn}}^2\sqrt{1 - \frac{1}{4\mathcal{E}_0^2}\frac{f_R^4}{f_B^2}\Omega_{\text{syn}}^4}$$
$$-\frac{\sigma\rho^2}{f_R}k(k-\kappa)\Omega_{\text{syn}}^2 = 0 \tag{26}$$

where $\Omega_{\text{syn}}(\bar{z}) = \left[-2\frac{f_B}{f_R^2}|\mathcal{E}(\bar{z})|\sin\psi_R\right]^{1/2}$.

NUMERICAL RESULTS

In this section we illustrate the numerical solutions to the sideband dispersion relation Eq. (26). We consider two distinct cases: one for hard x-ray 4.5-keV FEL based on LCLS-like parameters and the other for submilimeter range FEL [6]. Table 1 summarizes the relevant beam and undulator parameters for the sideband analysis. For simplicity we do not consider the undulator tapering but only focus on the difference of calculated sideband growth rate with or without longitudinal space charge effect.

Figure 1 shows the case for 4.5-keV hard x-ray FEL. One can barely see the difference with and without the longitudinal space charge effect. In contrast, such difference can be clear in the calculated sideband growth rate for the case of long-wavelength FEL. Shown in Fig. 2, it is found that the sideband growth rate with inclusion of space charge effect is reduced. This can be attributed by the fact that space charge effect will become important when the space-charge-induced plasma oscillation becomes evident within a synchrotron period. That is to say, the space charge effect becomes important when the Pierce parameter becomes larger.

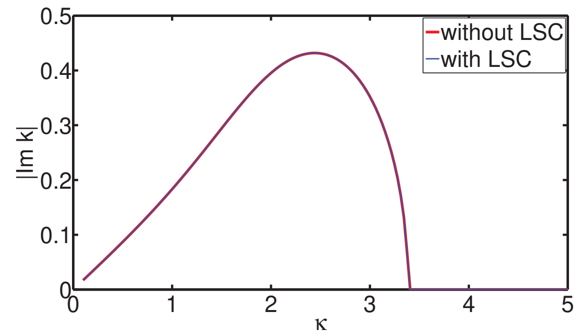

Figure 1: The sideband growth rate $|\text{Im}k|$ as a function of κ. Here $\Omega_{\text{syn},0} \approx 2.25$, $|\mathcal{E}| \approx 2.53$ and $f_B = f_R = 1$.

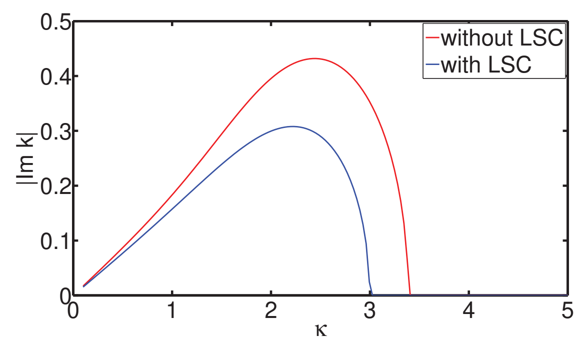

Figure 2: The sideband growth rate $|\text{Im}k|$ as a function of κ. Here $\Omega_{\text{syn},0} \approx 2.25$, $|\mathcal{E}| \approx 2.53$ and $f_B = f_R = 1$.

SUMMARY AND CONCLUSION

In the absence of space charge effect (i.e., $\sigma = 0$), we see that Eq. (26) is identical to the usual dispersion relation for tapered FEL [1,7]. We also note that the initial energy detuning (δ) does not appear in the dispersion relation although we started with the equations including the detuning term, confirming our previous conclusion that to first order the energy detuning does not contribute to the sideband instabilities. The space charge effect on the sideband instability growth is found to be negligible for short-wavelength FELs; however the sideband growth will be reduced in the long-wavelength FEL regime, where ρ becomes relatively large.

ISBN: 978-3-95450-184-7 doi:10.18429/JACoW-IPAC2018-THPMK054

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