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SPACE-CHARGE HAMILTONIAN WITH A SPACE COORDINATE AS INDEPENDENT VARIABLE

T. Planche*, P. M. Jung, S. D. Rädel, TRIUMF, Vancouver, Canada

Abstract

We present a version of the Low Lagrangian tailored to treat space-charge effects in particle accelerators: the Lagrangian is relativistic and uses a space coordinate as the independent variable. From this Lagrangian we obtain the corresponding Hamiltonian. From the Hamiltonian we obtain equations of motion for the 8 canonical variables, which can be plugged into a symplectic numerical integrator. We will finally discuss the possibility of numerically solving this problem using an explicit symplectic integrator.

INTRODUCTION

A Lagrangian for non-relativistic collisionless plasmas was proposed by Low [1]. Relativistic versions of this Lagrangian have been proposed for both the electrostatic [2] and the full electromagnetic [3] cases. In particle accelerators, the electrostatic approximation is generally sufficient as particles do not move at relativistic speeds with respect to each other. The electrostatic Lagrangian does not explicitly depend on the partial derivative of the scalar electric potential with respect to time ($\partial_t \phi$). This degeneracy makes it impossible to determine a momentum canonically conjugated to ϕ for a Lagrangian with time as the independent variable. To lift this degeneracy we propose to use a Lagrangian with a space coordinate as the independent variable.

LAGRANGIAN

For simplicity we restrict ourselves to the case of a beam moving straight in the z direction. We also assume that all particles move in the same forward direction (i.e. $\partial_z t > 0$). To simplify our notation we choose to work in a system of units where the speed of light, the vacuum permitivity and the vacuum permeability are all equal to 1. Starting from Ref. [2], and proceeding to a change of independent variable as described in Ref. [4], we obtain the following Lagrangian:

$$\begin{aligned} &L(\mathbf{X}, \mathbf{X}', \phi, \phi'; z) = \\ &- \iint \left(\hat{m}\sqrt{-1 - \mathbf{X}'^2} + t'\hat{q}\phi(\mathbf{X}) \right) \mathrm{d}^3 x_1 \mathrm{d}^3 p_1 \\ &+ \frac{\gamma_0(z)}{2} \int (\nabla\phi(\mathbf{x}_1))^2 \ \mathrm{d}^3 x_1 \,, \end{aligned} \tag{1}$$

where $\hat{q} = q \cdot f_1(\mathbf{x}_1, \mathbf{p}_1)$ is the charge density, $\hat{m} = m \cdot f_1(\mathbf{x}_1, \mathbf{p}_1)$ is the mass density, $\mathbf{X}(\mathbf{x}_1, \mathbf{p}_1) = (x, y, it)$, $\nabla\phi = \partial_x\phi + \partial_y\phi + \phi'$, primes denote partial derivatives with respect to z, and $\gamma_0(z)$ is the Lorentz factor of the beam centroid.

* tplanche@triumf.ca

Taking partial derivatives of L with respect to $\mathbf{X}'$ and ϕ' gives us the canonical momenta:

$$\begin{aligned} P_x &= \frac{\hat{m}x'}{\sqrt{t'^2 - x'^2 - y'^2 - 1}} \,, \\ P_y &= \frac{\hat{m}y'}{\sqrt{t'^2 - x'^2 - y'^2 - 1}} \,, \\ iE &= i\frac{\hat{m}t'}{\sqrt{t'^2 - x'^2 - y'^2 - 1}} - i\hat{q}\phi \,, \\ P_\phi &= \gamma_0 \phi' \,. \end{aligned} \tag{2}$$

The first three are the usual transverse canonical momenta and i times the "single-particle" total energy. The fourth momenta is what we were looking for.

HAMILTONIAN

Performing the Legendre transformation yields:

$$\begin{aligned} &H(\mathbf{X}, \mathbf{P}, \phi, P_\phi; z) = \\ &- \iint \sqrt{-\hat{m}^2 - P_x^2 - P_y^2 + (E - \hat{q}\phi(\mathbf{X}))^2} \, \mathrm{d}^3 x_1 \mathrm{d}^3 p_1 \\ &+ \frac{\gamma_0}{2} \int \left(\frac{P_\phi^2}{\gamma_0} - \partial_x\phi^2 - \partial_y\phi^2 \right) \mathrm{d}^3 x_1 \,, \end{aligned} \tag{3}$$

where $\mathbf{P} = (P_x, P_y, iE)$. The associated canonical Poisson bracket is: $\{F, G\} =$

$$\iint \frac{\delta F}{\delta \mathbf{X}} \frac{\delta G}{\delta \mathbf{P}} - \frac{\delta F}{\delta \mathbf{P}} \frac{\delta G}{\delta \mathbf{X}} \mathrm{d}^3 x_1 \mathrm{d}^3 p_1 + \int \frac{\delta F}{\delta \phi} \frac{\delta G}{\delta P_\phi} - \frac{\delta F}{\delta P_\phi} \frac{\delta G}{\delta \phi} \mathrm{d}^3 x_1, \tag{4}$$

where δ denote functional derivatives. Applying this Poisson bracket leads to the following 8 equations of motion:

$$\begin{aligned} x' &= \frac{P_x}{\sqrt{-\hat{m}^2 - P_x^2 - P_y^2 + (E - \hat{q}\phi)^2}} \,, \quad P_x' = -\hat{q}t'\partial_x\phi \,, \\ y' &= \frac{P_y}{\sqrt{-\hat{m}^2 - P_x^2 - P_y^2 + (E - \hat{q}\phi)^2}} \,, \quad P_y' = -\hat{q}t'\partial_y\phi \,, \\ it' &= i\frac{E - q\phi}{\sqrt{-\hat{m}^2 - P_x^2 - P_y^2 + (E - \hat{q}\phi)^2}} \,, \quad iE' = -\hat{q}t'\partial_{it}\phi \,, \\ \phi' &= \frac{P_\phi}{\gamma_0} \,, \quad P_\phi' = -\gamma_0(\partial_{xx}\phi + \partial_{yy}\phi) - \int t'\hat{q} \, \mathrm{d}^3 p_1 \,. \end{aligned} \tag{5}$$

There is no new physics in these equations: the first three pairs are similar to the ones obtained from the Courant-Snyder's Hamiltonian [5]. The last two equations combined

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together lead to:

$$\nabla^2 \phi = -\frac{\rho}{\gamma_0}, \tag{6}$$

where $\rho = \int t' \hat{q} \, \mathrm{d}^3 p_1$. One recognizes Poisson's equation with relativistic space dilatation.

At this point one could discretize the system – the phase space into macro particles, and the real space on a grid – and solve the equations of motion for all $(\mathbf{X}, \mathbf{P})$ of the macro particles and all (ϕ, P_ϕ) of the grid nodes using a numerical integrator. Since the equations of motion are all obtained from the same Hamiltonian it is tempting try to use a symplectic integrator. This integrator would, a priori, have to be implicit, since these equations (except the trivial one for ϕ') depend explicitly on both positions and momenta.

PARAXIAL TRANSVERSE HAMILTONIAN

For practical application to accelerators, it is essential to add to our Hamiltonian the contribution from external (focusing) forces. We choose to do it by adding a vector potential term $\mathbf{A}$. The Hamiltonian becomes:

$$\begin{aligned} &H(\mathbf{X}, \mathbf{P}, \phi, P_\phi; z) = \\ &\iint \left(\hat{q} A_z(\mathbf{X}) - \sqrt{-\hat{m}^2 + e^2 + p_x^2 + p_y^2} \right) \mathrm{d}^3 x_1 \mathrm{d}^3 p_1 \\ &+ \frac{\gamma_0}{2} \int \left(\frac{P_\phi^2}{\gamma_0} - \partial_x \phi^2 - \partial_y \phi^2 \right) \mathrm{d}^3 x_1 , \end{aligned} \tag{7}$$

where $e = E - \hat{q}\phi(\mathbf{X})$ and $p_{x,y} = (P_{x,y} - \hat{q} A_{x,y})/P_0$.

Let's now transform the longitudinal variables (it, iE) into $(\Delta t, -\Delta E)$, where:

$$\begin{aligned} \Delta t &= t - t_0 , \\ \Delta E &= E - E_0 , \\ E_0 &= \frac{\hat{m}\, t_0}{\sqrt{t_0'^2 - 1}} = \hat{m}\, \gamma_0 . \end{aligned} \tag{8}$$

For simplicity we will assume that $\mathbf{A}$ is time independent, which implies that E_0 is a constant. The generating function for this canonical transformation is:

$$F_2(t, \Delta E) = -\left(t - \int t_0(z)\, \mathrm{d}z \right) (E_0 + \Delta E) . \tag{9}$$

The new Hamiltonian is obtained by adding $\partial_z F_2$ to the Hamiltonian density under the first integral.

Restricting ourselves to the transverse motion, i.e. assuming a coasting beam with $\Delta E = 0$, and making the paraxial approximation, i.e. assuming that $P_x - \hat{q} A_x$, and $P_y - \hat{q} A_y$ are small compared to $P_0 = \sqrt{E_0^2 - m^2}$, yields:

$$\begin{aligned} &H_{\text{paraxial}}(x, P_x, y, P_y, \phi, P_\phi; z) = \\ &\iint \left(p_x^2 + p_y^2 + \frac{(\hat{m}\hat{q}\phi(\mathbf{X}))^2}{2 P_0^3} - \hat{q} A_z(\mathbf{X}) \right) \mathrm{d}^3 x_1 \mathrm{d}^3 p_1 \\ &+ \frac{\gamma_0}{2} \int \left(\frac{P_\phi^2}{\gamma_0} - \partial_x \phi^2 - \partial_y \phi^2 \right) \mathrm{d}^3 x_1 . \end{aligned} \tag{10}$$

Applying the Poisson bracket given in Eq. (4), we obtain the following equations of motion:

$$\begin{aligned} &x' = p_x , \; P_x' = -\hat{q} \left(\tau' \partial_x \phi + p_x \partial_x A_x + p_y \partial_x A_y + \partial_x A_z \right) , \\ &y' = p_y , \; P_y' = -\hat{q} \left(\tau' \partial_y \phi + p_x \partial_y A_x + p_y \partial_y A_y + \partial_y A_z \right) , \\ &\phi' = \frac{P_\phi}{\gamma_0} , \quad P_\phi' = -\gamma_0 (\partial_{xx}\phi + \partial_{yy}\phi) - \int \tau' \hat{q} \, \mathrm{d}^3 p_1 , \end{aligned} \tag{11}$$

where $\tau' = \hat{q}\hat{m}^2 \phi(x, y)/P_0^3$.

CONCLUSION AND FUTURE WORK

We have shown that it is possible to describe space-charge problems using a space coordinate as intependent variable. Our purpose is to exploit the similarities between Eq. (10) and the paraxial single-particle Hamiltonian in Ref. [6] to develop an explicit symplectic integrator to solve Eqs. (11). We are currently working on the implementation of a toy model to test the validity of this approach.

REFERENCES

[1] F. E. Low, A Lagrangian formulation of the Boltzmann-Vlasov equation for plasmas, Proceedings of the Royal Society of London. Series A, Mathematical and Physical Sciences 248 (1253) (1958) 282–287. http://www.jstor.org/stable/100602

[2] R. T. Glassey and J. Schaeffer, On symmetric solutions of the relativistic Vlasov-Poisson system, Communications in Mathematical Physics 101 (4) (1985) 459–473.

[3] B. A. Shadwick, A. B. Stamm, and E. G. Evstatiev, Variational formulation of macro-particle plasma simulation algorithms, Physics of Plasmas 21 (5) (2014) 055708.

[4] R. Broucke, On changes of independent variable in dynamical systems and applications to regularization, Icarus 7 (1-3) (1967) 221–232.

[5] E. Courant and H. Snyder, Theory of the alternating-gradient synchrotron, Annals of Physics 3 (1) (1958) 1 – 48. doi:http://dx.doi.org/10.1016/0003-4916(58)90012-5. http://www.sciencedirect.com/science/article/pii/0003491658900125

[6] Y. K. Wu, E. Forest, and D. S. Robin, Explicit symplectic integrator for s-dependent static magnetic field, Physical Review E 68 (4) (2003) 046502.