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SYMBOLIC PRESENTATION OF NONLINEAR DYNAMIC SYSTEMS IN TERMS OF LEGO-OBJECTS

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Abstract

In this paper we propose a symbolic representation of the solutions of the equations of evolution of dynamical systems in the framework of matrix formalism and Lie algebra for a number of elements of the accelerator (in particular, quadrupoles and octupoles) up to the 4-th order. The considered solutions present so-called Lego objects, which are included in the general scheme of the representation beam dynamics and used for its preliminary and computer modeling.

INTRODUCTION

Currently, most of the programs used for preliminary and computer modeling of accelerators are based on numerical methods. The most widely used packages are MAD [1], Cosy Infinity [2], MaryLie [3] and Comsol Multiphysics [4]. MAD, COSY Infinity and Comsol Multiphysics use phase variables to describe the beam using trajectory analysis. However, the problems of modeling involve the selection of optimal parameters for the systems under consideration, so the software to solve such problems should allow a quick change the parameters of the system.

Numerical methods can't cope with such a task, because any change in the parameters leads to a complete recalculation of the trajectories for all considered particles (10^{12}–10^{16} in beam). In addition, when we talk about cyclic systems, such as an accelerator ring, the calculations increase in proportion to the number of turns. Thus, for this problem, numerical methods will become very time-consuming.

The Marylee package, though based on symbolic methods, does not support one more prerequisite for providing the fastest solution to the optimization problems of complex systems – the Lego-object approach [5]. The ideology of this approach lies in the fact that the contribution of each element included in the finite system (for example dipoles, quadrupoles and octupoles, etc.) is considered independently from the rest of the system. The resulting solution for one or more elements can be substitute in the "right place". This provides the possibility of easy changes and replacement of elements in the simulation of the final nonlinear dynamic system. This work is an illustration of the application of a new approach to solving problems of preliminary and computer modeling of nonlinear dynamics based on symbolic computation, supported the ideology of Lego-objects, as well as allowing for taking into account the symplecticity of the system and is not limited only to trajectory analysis.

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THEORETICAL BASIS

The approach uses a matrix formalism combined with the Lie algebra [6]. Due to matrix formalism, it becomes possible to consider the whole beam at the same time, as well as to move from the consideration of coordinates and impulses to the consideration of more convenient for the analysis of quantities.

We should note that although character calculations are time-consuming and result in cumbersome formulas, we should realize this operation only once for each control object in the system. Using this approach, we can not only create the database with Lego-objects but also use it in the process of modeling the systems under consideration. We also can solve the real task of optimization is quickly enough by linking the corresponding Lego-objects and by a setting of the necessary parameters for this objects.

In this section we describe a schematic description of the theory of constructing a solution of a system of ordinary differential equations in an explicit form using perturbation theory. Consider the equation of motion in the form $d\mathbf{X}/ds = \mathbf{F}(\mathbf{X}, s)$, $\mathbf{F}(0, s) = 0$, where $\mathbf{X}$ is a vector of phase moments and the arbitrary analytic function $\mathbf{F}(\mathbf{X}, s)$ is defined within a neighborhood of $\mathbf{X} = 0$, $\mathbf{X} \in R^n$ and is measurable at $s \in R^n$. The following definition for the vector of phase moment is necessary: $\mathbf{X}^{[k]} = \underbrace{\mathbf{X} \otimes \ldots \otimes \mathbf{X}}_{k \text{ times}}$ is a vector of phase moments of the k-th order, representing the k-th Kronecker degree of the phase vector $\mathbf{X}$. Further $\mathbb{P}^{1k}(s)$ is a matrix of dimension $(n \times d[n, k])$, where $d[n, k] = \binom{n+k-1}{k}$, the elements of which are the derivatives of the components of the vector-valued function $\mathbf{F}(0, s) = 0$ of the k-th order. Note that $\mathbb{P}^{1k}(s)$ are matrices which include the control parameters of the system. Thus, taking into account the assumption that $\mathbf{F}(0, s) = 0$, we can write:

$$\frac{d\mathbf{X}(s)}{ds} = \sum_{k=1}^{\infty} \mathbb{P}^{1k}(s)\mathbf{X}^{[k]}(s), \quad \mathbf{X}(\mathbf{s_0}) = \mathbf{X}_0 \tag{1}$$

The solution of such a nonlinear system in the form of two-dimensional matrices, which can be calculated according to the algorithm presented in [7], can be represented in the form:

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$$\mathbf{X}(s|s_0) = \sum_{k=1}^{\infty} \mathbb{R}^{1k}(s|s_0)\mathbf{X}_0^{[k]}. \quad (2)$$

The matrices $\mathbb{R}^{1k}(s_0)$ contain coefficients for the corresponding orders of the elements of the vector $\mathbf{X}_0$. A finite cut-off of this infinite series can be defined which follows from the properties of the object under consideration.

Multi-rotation in cyclic machines leads to the need to preserve the integrals of motion: the law of conservation of energy, symplecticity and etc. The requirements for modern cyclic systems lead to the necessity of using nonlinear control fields up to some k-th order, which in turn leads to a violation of the symplectic property. This is a critical moment for this similar of tasks. In terms of the matrix formalism, the symplecticity condition can be ensured using the Jacobi matrix $\mathbb{M}(\mathbf{X}, s|s_0)$ of the transformation $M(\mathbf{X}, s|s_0; H)$ of the dynamical system as follows:

$$\mathbb{M}^*(\mathbf{X}, s|s_0)\mathbb{J}(\mathbf{X})\mathbb{M}(\mathbf{X}, s|s_0) = \mathbb{J}(\mathbf{X}), \quad (3)$$
$$\mathbb{M}(\mathbf{X}, s|s_0; M) = \mathbb{M}(\mathbf{X}, s|s_0) = \frac{\partial M(\mathbf{X}, s|s_0; H) \circ \mathbf{X}}{\partial \mathbf{X}^*},$$

where H is the Hamiltonian of the system.

Symplectic property can be restored using various methods for trajectory-based modeling, in particular, the methods of canonical transformations [8]. This approach is used for each individual trajectory of particles ($10^{12}-10^{16}$) and needs a lot of time. We propose making necessary corrections not in the trajectory, but in the matrices $\mathbb{R}^{1k}(s_0)$. Our approach is based on the correction of the elements of the transformation matrices $\mathbb{R}^{1k}(s_0)$ up to the desired order of nonlinearity [9]. As a result, the amount of computations performed much less than on-trajectory analysis. Let us note that the simplification procedure must be performed sequentially for all orders (from 1 to the k-th), because the elements of each current transformation matrix have a relationship with the elements of the previous ones.

SOLUTIONS OF SEVERAL ELECTROMAGNETIC ELEMENTS

In this paper, are presented solutions obtained for some basic electromagnet elements used in the construction of particle accelerators: the quadrupoles and the octupoles. Hamiltonians of these elements were obtained at [7]:

Hamiltonian for quadrupole

$$H = \frac{1}{8}(P_x^2 + P_y^2)^2 + \frac{K_1'}{2}x^2 yP_y - \frac{K_1''}{48}(x^2 + 6x^2y^2 - y^4),$$

Hamiltonian for octupole

$$H = \frac{1}{8}(P_x^2 + P_y^2)^2 + \frac{K_3}{24}(x^2 - 6x^2y^2 + y^4),$$

where $K_i = (q/cP_0)A_{1i}$, A_{1i} - vector potential, P_0 - momentum of an equilibrium particle, q - particle charge, c - speed of light.

The systems of differential equations describing to the dynamics of the elements under consideration have the form:

systems for quadrupole

$$\begin{cases} x'(t) = \frac{1}{2}P_x(P_x^2 + P_y^2), \\ P_x'(t) = -xyP_yK_1' + \frac{1}{48}(2x + 12xy^2)K_1'', \\ y'(t) = \frac{1}{2}P_y(P_x^2 + P_y^2) + \frac{1}{2}x^2yK_1', \\ P_y'(t) = \frac{1}{48}(12x^2y - 4y^3)K_1'', \end{cases} \quad (4)$$

systems for octupole

$$\begin{cases} x'(t) = \frac{1}{2}P_x(P_x^2 + P_y^2), \\ P_x'(t) = -\frac{1}{24}(2x + 12xy^2)K_3, \\ y'(t) = \frac{1}{2}P_y(P_x^2 + P_y^2), \\ P_y'(t) = -\frac{1}{24}(12x^2y - 4y^3)K_3. \end{cases} \quad (5)$$

The solutions obtained by the considered method for visualization of the representation in the form of phase portraits after some simplifications. Recall that the reference trajectory is usually located on the plane. The procedure of simplifying is carried out by correcting the least important elements of the matrix $\mathbb{R}^{1k}(s_0)$ (for less distortion of the overall result). We corrected the coefficients for y and P_y to obtain the solutions presented in this work. Let us now turn to an illustration of the solutions. The phase portraits of quadrupole and dipole solutions are constructed for identical initial parameters:

$$x_0 = 0, \; y_0 = 0, \; P_{x0} = 1, \; P_{y0} = 1. \quad (6)$$

Figure 1 shows the phase portraits of the quadrupole solution for different parameter values. As can be seen from Fig. 1, when $K_1' = 0$ and $K_1'' = 2$ (Fig. 1a), the phase portrait is closed and has the circle shape. The change K_1'' leads to a gap at phase portrait (Fig. 1b), the variation of K_1' to a change in shape to a spiral (Fig. 1c).

Phase portraits of the octupole solution are represented in Fig. 2. By varying parameter K_3, we can conclude that for $K_3 = -1$, the phase portrait for the octupole has the circle shape (Fig. 2a), change K_3 leads to changes shape to a spiral (Fig. 2b).

Let us note that the solutions illustrated above are obtained in symbolic form, which allows using them as required the case of changing the system parameters. For the most convenient use, we plan to put them in the corresponding database with Lego-objects. Working with the database of such objects is carried out according to the following scheme (see Fig. 3).

The simulated accelerator is presented using free spaces, dipoles, quadrupoles, octupoles, and other control elements. For each of these objects, one can calculate the corresponding solutions and place in the appropriate database in advance (as a family of Lego-objects).The general control system defining the evolution of a system is made by a combination of objects from the database. To solve a specific problem, it is only necessary to substitute the values of the control elements parameters and perform the necessary numerical

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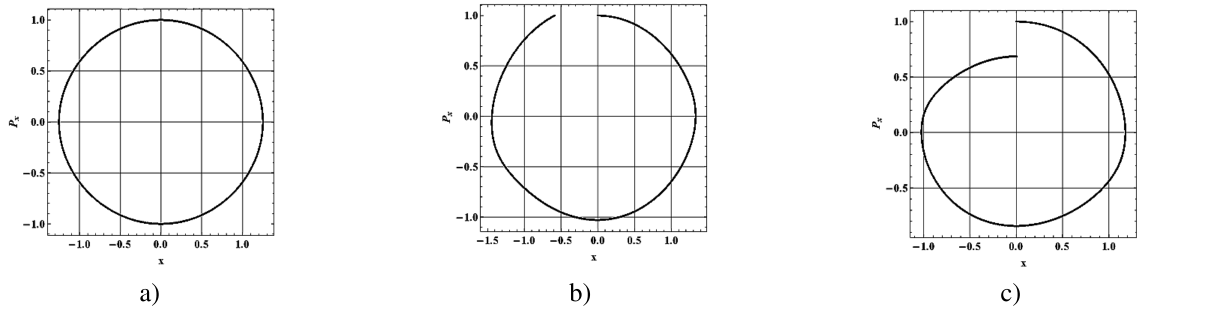

Figure 1: Phase portraits of the quadrupole symplectic solution with different parameters: a) $K_1' = 0$, $K_1'' = 2$, b) $K_1' = 0$, $K_1'' = 3$, c) $K_1' = 0.4$, $K_1'' = 2$.

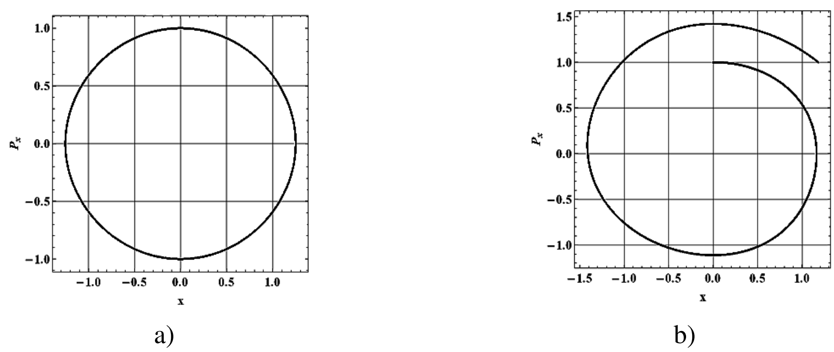

Figure 2: Phase portraits of the octupole symplectic solution with different parameters: a) $K_3 = -1$, b) $K_3 = 0$.

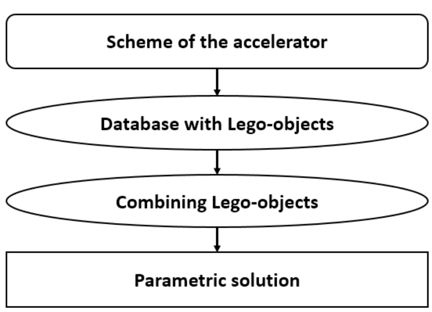

Figure 3: The diagram illustrates the operation of the Lego-objects database.

calculations and obtaining solutions for the simulated system.

CONCLUSION

In this paper, was submitted a method for symbolic solving differential equations of evolution of dynamical systems in the matrix framework and Lie algebra paradigm for quadrupole and octupole up to 4-th order. As an illustration of the results of the approach, we present phase portraits of solutions for quadrupole and octupole with a brief analysis. We note that all solutions can be obtained in some symbolical forms, which allows us obtaining the numerical solutions for varying parameters on demand. The resulting symbolic solutions will be placed in the special database (for Lego-objects) to be able to combine objects and obtain final solutions depending on the simulated accelerator scheme. This database is planned to be used for modeling within the framework of the NICA accelerator project.

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