MODE COUPLING THEORY IN COLLISIONS WITH A LARGE CROSSING ANGLE

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Abstract

We discuss a novel coherent beam-beam instability in collisions with a large crossing angle. The instability appears in the correlated head-tail motion of the two colliding beams. Cross wake force is introduced to represent the head-tail correlation between colliding beams. The cross wake force is localized at the collision point. Mode coupling theory based on the cross wake force is developed. Collision scheme with a large crossing angle is being very popular in design of electron positron collider. In SuperKEKB project, a collision with a large crossing angle is performed to boost the luminosity 0.8×10^{36} cm^{-2}s^{-1}. Future circular collider, FCC is also designed with a large crossing angle. Strong-strong simulations have shown a strong coherent head-tail instability, which can limit the performance of proposed future colliders. The mode coupling theory using the cross wake force explains the instability. The instability may affect all colliders designs based on the crab waist scheme.

INTRODUCTION

A cross wake force induced at beam-beam collision with a large crossing angle causes correlation between colliding beams. A correlated head-tail instability between the beams have been studied by mode coupling theory using the cross wake force [1]. Detailed studies for the head-tail instability had been done using particle tracking and eigen-mode analysis in Ref [2]. Main advance from Ref. [2] is to take into account of z dependent tune shift in the beam-beam collision. The cross wake force induces tune shift depending on z. In Ref. [2], a constant tune shift, which is an averaged value of the beam distribution, was considered. Mode coupling theory of two beams under z dependent tune shift is discussed in this paper.

CROSS WAKE FORCE

In collision with a large crossing angle, two bunches with betatron amplitudes $x(z)$ and $x(z')$ interact along the progress of collision at $s = (z - z')/2$. The momentum kick at z position in positron bunch is expressed by [2]

$$\Delta p_x^{(+)}(z) = -\frac{N r_e}{\gamma} \int F_x[x^{(+)}(z) - x^{(-)}(z') + \Delta x] \rho^{(-)}(z') dz'. \tag{1}$$

The offset of the collision between longitudinal positions z and z' is $\Delta x = 2s\theta_c = (z - z')\theta_c$, where θ_c is the half crossing angle. Expanding for x, cross wake force $W^{(\pm)}(z - z')$,
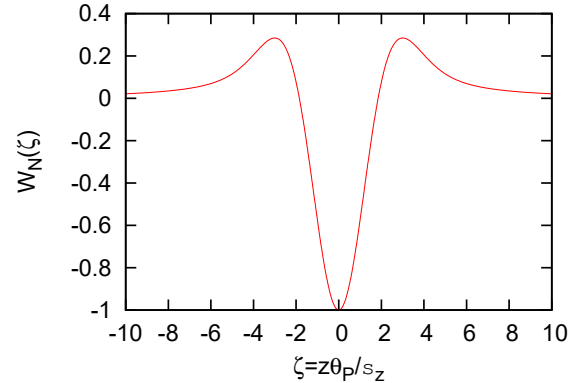
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Figure 1: Normalized cross wake force, $W_N(\zeta) = -1 + \sqrt{\pi}\zeta/2 \, \text{Im}\, w(\zeta/2)$, where $\zeta = \theta_P z/\sigma_z$.

which represents correlation of two bunches, is introduced as follows,

$$\Delta p_x^{(\pm)}(z) = -\int_{-\infty}^{\infty} W_x^{(\pm)}(z - z') \rho^{(\mp)}(z') x^{(\mp)}(z') dz'$$
$$+ \int_{-\infty}^{\infty} W_x^{(\pm)}(z - z') \rho^{(\mp)}(z') dz' x^{(\pm)}(z). \tag{2}$$

The first term of RHS gives correlation between colliding two beams. Second term gives the momentum change linearly depends on $x(z)$: that is, tune shift as function of z.

The cross-wake force for e$^{\pm}$ beams can be expressed by

$$W_x^{(\pm)}(z) = -\frac{N^{(\mp)} r_e}{\gamma^{(\pm)}} \frac{\partial F_x(\Delta x)}{\partial x} = \frac{N^{(\mp)} r_e}{\gamma^{(\pm)} \bar{\sigma}_x^2} W_N(\zeta). \tag{3}$$

The normalized wake force is defined by

$$W_N(\zeta) \equiv -\bar{\sigma}_x^2 \frac{\partial F_x(\Delta x)}{\partial x} = -1 + \sqrt{\pi}\zeta/2 \, \text{Im}\, w(\zeta/2)$$
$$= -1 + \zeta e^{-\zeta^2/4} \int_0^{\zeta/2} e^{u^2} du, \tag{4}$$

where $\zeta = \theta_P z/\sigma_z$. $\bar{\sigma}_x^2 = (\sigma_x^{(+)2} + \sigma_x^{(-)2})/2$. $\theta_P = \theta_c \sigma_z/\bar{\sigma}_x$.

Figure 1 shows the normalized cross-wake force. This wake force is significant in the area several times the size of $\bar{\sigma}_z/\theta_P = \bar{\sigma}_x/\theta_c$.

MODE COUPLING THEORY FOR THE CROSS WAKE FORCE

We focus motion of the dipole amplitude in the longitudinal phase space (J, ϕ),

$$x(J, \phi), \qquad p_x(J, \phi). \qquad (5)$$

The longitudinal variables are the arrival time advance $z = ct - s$ and the momentum deviation $p_z = \Delta p/p_0$ normalized by total momentum. The synchrotron amplitude J and phase ϕ are related to the variables as $z = \sqrt{2\beta_z J}\cos\phi$ and $p_z = \sqrt{2J/\beta_z}\sin\phi$. Beam density distribution in the phase space is assumed to be Gaussian only depending on J as,

$$\psi(z, p_z) = \frac{1}{2\pi\varepsilon_z}\exp(-J/\varepsilon_z). \qquad (6)$$

The density distribution and dipole amplitude for z are given by

$$\rho(z) = \int dp_z \psi(J) \quad x(z) = \frac{1}{\rho(z)}\int x(J, \phi)\psi(J)dp_z. \qquad (7)$$

The momentum kick Eq. (2) is expressed by the longitudinal phase space variables

$$\Delta p_x^{(\pm)}(J, \phi) =$$
$$- \int_{-\infty}^{\infty} W_x^{(\pm)}(z - z')\psi^{(\mp)}(J', \phi')x^{(\mp)}(J', \phi')dJ'd\phi'$$
$$+ \int_{-\infty}^{\infty} W_x^{(\pm)}(z - z')\psi^{(\mp)}(J', \phi')dJ'd\phi' x^{(\pm)}(J, \phi). \qquad (8)$$

This formula can be regarded as a matrix form. The synchrotron phase space is discretized by J_i and ϕ_j, where $i = 1, n_J$ and $j = 1, n_\phi$. The integral is represented by summation with a small step $\Delta J, \Delta\phi$.

We consider σ or π mode for the motion of two beams. The dipole amplitudes are related as $x^{(+)}(z) = x^{(-)}(z)$ or $x^{(+)}(z) = -x^{(-)}(z)$, respectively. The cross wake force in Eq. (8) can be treated as an ordinary single bunch wake force.

A vector $(x_{ij} = x(J_i, \phi_j), p_{ij} = p_x(J_i, \phi_j))$ with the length $2n_J n_\phi$ is transferred by following matrix

$$M_W = \begin{pmatrix} 1 & 0 \\ \beta_x W_{ij,i'j'} & 1 \end{pmatrix}, \qquad (9)$$

where

$$W_{ij,i'j'} = \Big[\mp W(z_{ij} - z_{i'j'})\psi_{i'}$$
$$+ \sum_{i''j''} W(z_{ij} - z_{i''j''})\psi_{i''}\delta_{i,i'}\delta_{j,j'}\Big]\Delta J\Delta\phi. \qquad (10)$$

Upper $(-)$ and lower $(+)$ signs are for σ and π modes, respectively.

Synchrotron/betatron motion is expressed by matrix transformation as follows,

$$M_{0,ij,i'j'} = \delta_{i,i'}\delta_{j-1,j'}\begin{pmatrix} \cos\mu_x & \sin\mu_x \\ -\sin\mu_x & \cos\mu_x \end{pmatrix}. \qquad (11)$$

where the synchrotron tune is approximated to be inverse of an integer, $\nu_s = 1/n_\phi$. The synchrotron motion is represented by transformation j to $j + 1$.

Transformation for a revolution is expressed by the multiplication of the two matrices,

$$\begin{pmatrix} x_{ij} \\ p_{ij} \end{pmatrix} = (M_W M_0)_{ij,i'j'}\begin{pmatrix} x_{i'j'} \\ p_{i'j'} \end{pmatrix}. \qquad (12)$$

The revolution matrix $M_W M_0$ is symplectic. Eigenvalues of the matrix determines stability of the dipole amplitude of the colliding beams. The eigenvalues are written as $\exp(\pm i\mu)$. When μ is imaginary, the motion is unstable. The unstable motion occurs when integer/half integer resonances or mode coupling; namely a threshold exists for the instability.

We show eigen-mode results for beam parameter of FCCee-Z [3]. The eigenvalues (λ_{ij}) are calculated for changing the bunch population. Eigentunes and growth rates are obtained by $\nu_{ij} = \text{atan}(\text{Im}\lambda_{ij}/\text{Re}\lambda_{ij})/2\pi$ and $g = \log|\lambda_{ij}|$. Figure 2 shows tune (top) and growth rate per turn (bottom) for σ mode as function of the beam intensity normalized by the design values. The threshold is $N/N_0 = 0.23$, two times higher than that without the tune shift term [2]. The threshold agrees with that for constant tune shift [2]. Figure 3 shows growth rate for π mode.

Figure 2: Eigenvalues of σ modes for FCCee-Z HiLum as function of beam intensity. Top and bottom are tune and growth rate, respectively..

The method is extended to two beam motion using twice longer vector $(x_{ij}^{(+)}, p_{ij}^{(+)}, x_{ij}^{(-)}, p_{ij}^{(-)})$. The transfer matrix for
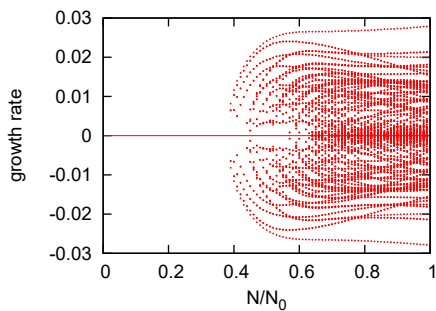
Figure 3: Eigenvalues (growth rate) of π modes for FCCee-Z HiLum as function of beam intensity..

the momentum kick of Eq. (2) is expressed by

$$
M_W = \begin{pmatrix}
1 & 0 & 0 & 0 \\
\beta_x^{(+)} K_{ij,i'j'}^{(+)} & 1 & \sqrt{\beta_x^{(+)}\beta_x^{(-)}} W_{ij,i'j'}^{(+)} & 0 \\
0 & 0 & 1 & 0 \\
\sqrt{\beta_x^{(+)}\beta_x^{(-)}} W_{ij,i'j'}^{(-)} & 0 & \beta_x^{(-)} K_{ij,i'j'}^{(-)} & 1
\end{pmatrix},
$$
(13)

where

$$
W_{ij,i'j'}^{(\pm)} = -W^{(\pm)}(z_{ij} - z_{i'j'})\psi_{i'}^{(\mp)}\Delta J \Delta\phi,
$$
(14)

$$
K_{ij,i'j'}^{(\pm)} = \sum_{i''j''} W^{(\pm)}(z_{ij} - z_{i''j''})\psi_{i''}^{(\mp)}\delta_{ii'}\delta_{jj'}\Delta J \Delta\phi.
$$
(15)

Figure 4 shows tune (top) and growth rate (bottom) for the two beam analysis. The growth rate agrees with merged one, in which those given by single beam method for σ and π modes.

SUMMARY AND DISCUSSIONS

The beam-beam collision with a large crossing angle induces a cross wake force which causes a correlation between two beam motion and z dependent tune shift. There are several mode expansion methods to study stability of the dipole amplitude distribution of two beams. In this paper, the dipole amplitude distribution is mapped in the discretized longitudinal phase space. This method considering z dependent tune shift works well for the case satisfying the transparency condition, where the synchrotron tune is approximated to be $1/n$.

For breaking the transparency condition, the matrix M_W is not symplectic. Eigenvalues are kept to be pairs of $\exp(\pm i\mu)$,
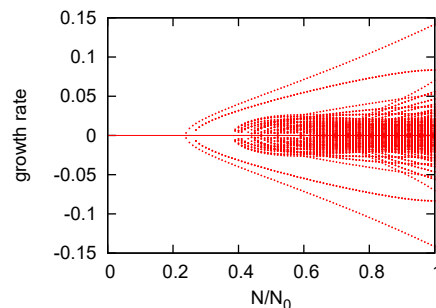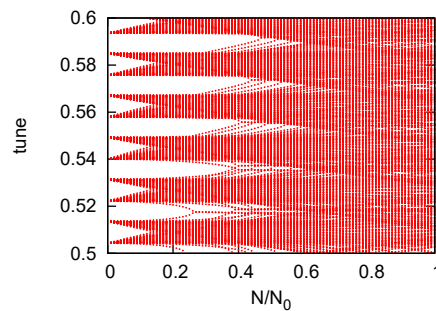

Figure 4: Eigenvalues of two beam modes for FCCee-Z HiLum as function of beam intensity. Top and bottom are tune and growth rate, respectively..

when two beams have an equal ν_s. The submatrices $W_{iji'j'}^{(\pm)}$ which give e$^\pm$ beams correlation are not square, when they have unequal ν_s. Eigenvalues are not pair of $\exp(\pm i\mu)$. The growth rate scatters for the tune. The result seems to be poor in reliability for unequal ν_s case.

In Ref. [2] the dipole amplitude distribution is expanded azimuthal and radial modes. Synchrotron tune can be chosen arbitrary in the mode expansion. Symmetry of matrix is broken for breaking transparency condition and unequal ν_s. We have not yet established the eigen mode analysis method to discuss the mode coupling for the cross wake force.

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