Overlapping Synchrotron Sideband Resonances*

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Abstract

The synchrotron sideband spin resonances are shown to arise from the kinematic effect of spin phase modulation with resonance strengths proportional to that of the primary spin resonance. We develop a method to analyze overlapping spin resonances and apply it to fit data recently obtained from polarized beam experiments at the IUCF Cooler Ring. The implication of our analyses is that synchrotron sidebands can only be corrected by correcting its principle resonance. Furthermore, the effect of synchrotron sidebands in proton synchrotrons is to change the resonance phase without affecting the magnitude of the strength.

1 INTRODUCTION

In synchrotrons, strong quadrupole fields are needed to focus the beam. Those particles moving off-center vertically in quadrupoles will experience horizontal fields, which can perturb the spin vector away from the vertical axis. Using the Thomas-BMT equation [1], the spin resonance strength is given by the Fourier amplitude of the spin perturbing fields in synchrotrons [2, 3], i.e.

$$\epsilon_K = \frac{1}{2\pi}\oint\left[(1+G\gamma)\frac{\Delta B_x}{B\rho}+(1+G)\frac{\Delta B_\parallel}{B\rho}\right]e^{iK\theta}ds, \tag{1}$$

where θ is the orbital bending angle, ΔB_x is the radial perturbing field, $\Delta B_\parallel$ is the longitudinal perturbing field, and $B\rho$ is the magnetic rigidity of the beam. In synchrotrons, there is little or no longitudinal field and the transverse radial field arises mainly from dipole rolls and the vertical displacement in quadrupoles. Neglecting the effect of dipole rolls, the radial perturbing field is given by

$$\Delta B_x = \frac{\partial B_z}{\partial x}z + \mathrm{HOM}, \tag{2}$$

where HOM stands for higher order multipoles. Spin resonance tunes are generally given by

$$K = n + k\nu_z + \ell\nu_x + m\nu_{syn}, \tag{3}$$

where k, ℓ, m, n are integers.

Defining a 2-component spinor Ψ with $\vec{S} = \langle\Psi|\vec{\sigma}|\Psi\rangle$, the Thomas-BMT equation can be casted into the spinor equation:

$$\frac{d\Psi}{d\theta} = -\frac{i}{2}\begin{pmatrix} G\gamma & -\xi \\ -\xi^* & -G\gamma \end{pmatrix}\Psi, \tag{4}$$

where ξ can be expanded in Fourier series:

$$\xi = \sum \epsilon_K e^{-iK\theta}. \tag{5}$$

Here ϵ_K is the resonance strength given by Eq. (1).

2 SYNCHROTRON SIDEBAND RESONANCES

We consider the spin equation of motion for a single primary spin resonance:

$$\frac{d\Psi}{d\theta} = -\frac{i}{2}\begin{pmatrix} G\gamma & -\epsilon_K e^{-iK\theta} \\ -\epsilon_K^* e^{iK\theta} & -G\gamma \end{pmatrix}\Psi, \tag{6}$$

where K is the resonance tune and ϵ_K is the resonance strength. For an off momentum particle with *linear* synchrotron motion, the spin tune is given by $G\gamma = G\gamma_0(1+\frac{\beta^2\Delta p}{p_0})$. Transforming the spinor wave function into the spin precessing frame with

$$\Psi = e^{-\frac{i}{2}\int_0^\theta G\gamma d\theta\sigma_3}\Psi_I, \tag{7}$$

the spin precessing phase for an off-momentum particle becomes

$$\int_0^\theta G\gamma d\theta = G\gamma_0\theta + \frac{\beta^2 G\gamma_0}{\nu_{syn}}\hat{a}\sin\nu_{syn}\theta. \tag{8}$$

In particular, we note that the spin precessing phase has been greatly enhanced by the smallness of the synchrotron tune. The effective spin phase modulation amplitude is $g = \frac{\beta^2 G\gamma_0}{\nu_{syn}}\hat{a}$.

Expanding the spin precessing phase in Fourier harmonics, the effective resonance driving term in the spinor equation becomes

$$\epsilon_K e^{-i(K\theta - \int G\gamma d\theta)} = \sum_{-\infty}^{\infty}\epsilon_K J_m(g)e^{-i(K-G\gamma_0-m\nu_{syn})\theta}. \tag{9}$$

If the condition $|\epsilon_K J_m(g)| < \nu_{syn}$ is satisfied, each synchrotron sideband behaves as an isolated resonance with

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resonance strength $\epsilon_K J_m(g)$, where $m = 0, \pm 1, \pm 2, \cdots$, i.e. the off-momentum particle experiences spin resonances at *all* synchrotron sidebands.

The physics of synchrotron sidebands can be understood as follows. The spin phase modulation due to a linear synchrotron motion can generate many sidebands around the spin tune. If one of the sidebands falls on the principle resonance, the spin is strongly perturbed. Thus the resonance strength of synchrotron sidebands is proportional to the strength of the principle resonance. Because the synchrotron tune is relatively small, the particle stays at the spin resonance condition for a long time, and therefore the effect of the spin resonance is particularly enhanced.

3 EFFECT OF OVERLAPPING RESONANCES

We consider a simple model of overlapping resonances with

$$\xi = \epsilon_1 e^{-iK_1\theta} + \epsilon_2 e^{-iK_2\theta}, \tag{10}$$

where ϵ_1, ϵ_2 and K_1, K_2 are resonance strengths and resonance tunes respectively. Let $\Delta = K_2 - K_1$ be the spacing of these two spin resonances. We can classify multiple resonances into three categories:

1. Isolated resonances with $|\Delta| \gg \max(\epsilon_1, \epsilon_2)$,
2. Overlapping resonances with $|\Delta| \ll \min(\epsilon_1, \epsilon_2)$,
3. Nearly overlapping resonances with $|\Delta| \geq \max(\epsilon_1, \epsilon_2)$.

The first case has been extensively studied in Ref. [2]. For the second case with overlapping spin resonances, the effective spin resonance strength becomes the arithmetic sum of all resonances [3]. Thus the effective resonance strength for synchrotron sidebands of proton synchrotrons becomes

$$\epsilon_{\text{eff}} \approx \epsilon_K \sum J_m(g) e^{im\nu_{syn}\theta_0} = \epsilon_K e^{ig\sin\nu_{syn}\theta_0}. \tag{11}$$

This means that the synchrotron motion merely changes the phase of the principle resonance strength without affecting its magnitude. We now turn to the case of nearly overlapping resonances.

In the nearly overlapping resonance regime with $\Delta \geq \max(\epsilon_1, \epsilon_2)$, the Froissart-Stora formula may not be applicable. We will analyze the two resonance model of Eq. (10) as follows.

Let us transform the spinor equation onto the resonance precession frame of K_1, i.e.

$$\Psi_{K1} = e^{i\frac{1}{2}K_1\theta\sigma_3}\Psi \tag{12}$$

and obtain

$$\frac{d\Psi_{K1}}{d\theta} = \frac{i}{2}\lambda_1(\hat{n}_1\cdot\vec{\sigma})\Psi_{K1} + \frac{i}{2}\begin{pmatrix} 0 & \epsilon_2 e^{-i\Delta\theta} \\ \epsilon_2^* e^{i\Delta\theta} & 0 \end{pmatrix}\Psi_{K1}, \tag{13}$$

where $\lambda_1 = \sqrt{\delta_1^2 + |\epsilon_1|^2}$, $\delta_1 = K_1 - G\gamma$, $\hat{n}_1 = \frac{1}{\lambda_1}(\delta_1\hat{e}_z + \epsilon_{1r}\hat{e}_x - \epsilon_{1i}\hat{e}_s)$, and $\epsilon_1 = \epsilon_{1r} + i\epsilon_{1i}$. The first term of Eq. (13) describes the precession of any arbitrary polarization vector around the spin closed orbit $\hat{n}_1$, which precesses around the vertical axis with the tune K_1. The second term describes the perturbation due to the nearby resonance K_2. If $|\delta_1 + \Delta| \gg |\epsilon_2|$, then the resonance K_2 can be treated perturbatively. We, however, will discuss the situation when $\delta_1 + \Delta \approx 0$ with $|\Delta| \geq \max(\epsilon_1, \epsilon_2)$.

The transformation of the spinor into the spin precession frame of K_1 can be accomplished with

$$\tilde{\Psi}_{K1} = e^{-\frac{i}{2}\lambda_1\theta\hat{n}_1\cdot\vec{\sigma}}\Psi_{K1}. \tag{14}$$

If $|\delta_1| \approx |\Delta| > |\epsilon_1|$, the spinor can similarly be transformed into the K_2 resonance frame with

$$\Psi_{K2} = e^{\frac{i}{2}\delta_2\theta\sigma_3}\tilde{\Psi}_{K1}.$$

Including the effect of the second resonance at K_2, the surviving polarization is

$$\vec{S}_{K1K2} = \frac{\delta_1^2}{\lambda_1^2} \times \frac{\delta_2}{\lambda_2}\hat{n}_2. \tag{15}$$

Thus the vertical polarization of nearly overlapping spin resonances can be expressed as the product of the projection of each resonance, i.e.

$$S_z = \prod_i \frac{\delta_i^2}{\lambda_i^2}, \tag{16}$$

and the radial polarization is given by

$$S_r = \sum_i \prod_{j\neq i} \frac{\delta_j^2}{\lambda_j^2} \times \frac{\delta_i|\epsilon_i|}{\lambda_i^2}\cos(K_i\theta + \chi_i). \tag{17}$$

The vertical and the horizontal polarization of the beam can then be obtained by *averaging* S_z and S_r over the particle distribution of the beam.

As the polarized beam travels around the ring, its polarization vector precesses about the vertical axis with a spin phase advance $K\theta$. Since θ advances by 2π every revolution, the radial polarization observed at one location in the ring will oscillate with a zero average if all resonance tunes K_i are not integers. On the other hand, if one of the resonances is an imperfection resonance, where K_i is an integer, the spin closed orbit is stationary and the radial polarization may not be zero.

4 DATA ANALYSIS

Synchrotron sidebands around an imperfection resonance have been observed in the spin dynamics experiments at the IUCF Cooler Ring [7]. Figure 1 shows the measured vertical and radial polarization vs the longitudinal field strength of the compensating solenoids at the IUCF Cooler Ring [7]. Vertically polarized protons at 104.5 MeV with 77% polarization were injected into the Cooler Ring and the radial and the vertical components of the beam polarization were measured as a function of the compensating solenoidal field

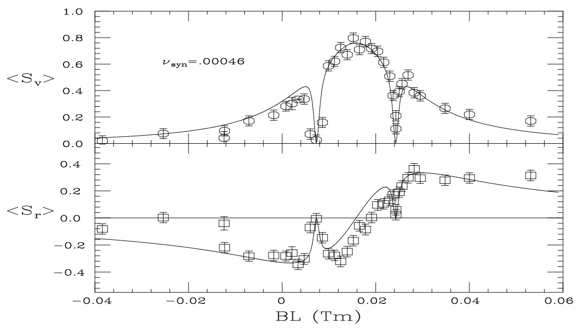

Figure 1: The vertical and radial polarization measured at the IUCF Cooler Ring for 104.5 MeV polarized protons is plotted as a function of the longitudinal transverse field error BL in T-m. When the spin tune was equal to the synchrotron tune, beam depolarization was observed. The solid line was obtained with a synchrotron amplitude $\hat{a} = 0.001$.

at the cooling section. The vertical polarization was found to be maximum at $B_{\|}L = 0.0158$ Tm, which corresponded to a fully compensated solenoidal field for spin motion.

The particle $G\gamma$ value for this experiment was 1.9925. Because of the vertical orbit bump at the electron cooling section, the spin precession tune was shifted upward by about 0.0035 [7]. Thus the spin tune of the beam was $\nu_s = 1.9960$. In the presence of the solenoidal field, the perturbed spin tune Q_s for an otherwise perfect synchrotron is given by

$$\cos\pi Q_s = \cos[\pi\nu_s]\cos\frac{\chi}{2}, \tag{18}$$

and the spin closed orbit vector $\hat{n}_1 = (n_{1x}, n_{1s}, n_{1z})$ is given by

$$n_{1x} = \frac{-1}{\sin\pi Q_s}\sin[\nu_s(\pi-\theta)]\sin\frac{\chi}{2}, \tag{19}$$

$$n_{1s} = \frac{1}{\sin\pi Q_s}\cos[\nu_s(\pi-\theta)]\sin\frac{\chi}{2}, \tag{20}$$

$$n_{1z} = \frac{1}{\sin\pi Q_s}\sin[\pi\nu_s]\cos\frac{\chi}{2}, \tag{21}$$

where θ is the orbital angle between the observation point and the solenoid ($\theta = 60°$ at the IUCF Cooler Ring), χ is the spin kick angle from the solenoid, i.e. $\chi = (1+G)\frac{\Delta B_{\|}L}{B\rho}$, where $\Delta B_{\|}L$ is the integrated solenoidal field error, and $B\rho$ is the beam rigidity. A larger spin precession angle χ will cause a larger deviation of the perturbed spin tune Q_s from an integer and the spin closed orbit also tilts further away from the vertical axis.

The solid line shown in Fig. 1 is the theoretical fit using Eqs. (16) and (17) with a type-3 snake tune shift of +0.0035 and a synchrotron tune 0.0046 with synchrotron amplitude $\hat{a} = \frac{2\nu_{syn}}{|\eta|B_f} \approx 0.001$, where $|\eta| \approx 0.76$ is the phase slip factor and $B_f \approx 10$ is the bunching factor. The resulting g is about 0.1. We note that the radial polarization is slightly shifted from the zero crossing point at a fully compensated solenoidal field. In an earlier study [7], we found that the regular imperfection resonance at $\nu_s = 2$ with resonance strength of the order of 0.0008 could give rise to a shift and asymmetry in the radial polarization. The solid line provides a good description of the synchrotron sidebands with only the single parameter of synchrotron amplitude $\hat{a}$. Without the kinematic enhancement, the resonance width would be too small to explain the data.

5 CONCLUSION

We have found that the spin synchrotron sideband resonances arise from the kinematic spin phase modulation, and that the strengths of these sidebands are proportional to that of the primary resonance. In the overlapping resonance regime, the spin resonances can be combined into a single resonance with an effective strength, which depends on the relative phase of each resonance. The effective spin resonance strength, including all synchrotron sidebands in proton accelerators, is equal to its principle resonance strength with a phase shift. The model for nearly overlapping spin resonances has successfully been used to analyze polarization data taken at the IUCF Cooler Ring.

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