THE EXPLICIT FORM FOR LIE TRANSFORMATIONS

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Abstract

This paper [1] presents a method of obtaining of explicit forms for Lie transformations which are widely used in particle physics problems. This approach is based on the matrix formalism for Lie algebraic tools. All calculations are realized in symbolic form with the help of computer algebra codes (*REDUCE* or *MAPLE* codes). The closed form of Lie transformations takes into account the intrinsic properties of the Lie transformations, for example, the property of symplecticity for Hamiltonian systems. The module which creates these closed forms for nonlinear systems is one of parts of a protype of an expert system which is created for beam line systems simulating.

1 BASIC DEFINITIONS AND EQUATIONS

1.1 Introduction

Beam line systems are usually described by nonlinear motion equations. The corresponding maps – Lie transformations – can be calculated with the help of so called Lie algebraic tools (for Hamiltonian systems see [1]). Usually Taylor expansions of these maps are used. However in this case we encounter with two problems. One is the loss of calculation accuracy and the other is the quality properties loss. The purpose of this paper is to present a new approach to evaluation of explicit forms for Lie transformations. This approach has the advantage that it is based on linear algebra algorithms, which are very well developed.

1.2 The matrix formalism

Further we will consider differential equations of motion in the general form

$$\frac{dX}{dt} = F(X,t), \tag{1}$$

where the function $F(X,t)$ can be represented as a Taylor expansion and we can write

$$\frac{dX}{dt} = \sum_{k=0}^{\infty} \mathbf{P}^{1k}(t) X^{[k]}. \tag{2}$$

Here $X^{[k]} = \underbrace{X \otimes \ldots \otimes X}_{k-times}$ is the so called Kronecker power of k-order for a phase vector $X = \{x_1, \ldots, x_n\}$, $\mathbf{P}^{1k}(t)$ are matrices containing Taylor expansion coefficients. The solution of the Eq.(2) can be written in the form

$$X(t) = X(X_0; t|t_0) = \sum_{k=0}^{\infty} \mathbf{M}^{1k}(t|t_0) X_0^{[k]}, \tag{3}$$

where $X_0 = X(t_0)$ is an initial phase vector and $\mathbf{M}^{1k}(t|t_0)$ are solution matrices. In the previous papers (f.e. see [2, 3, 4, 5]) basic features of this approach were demonstrated and different examples of applications were described. According to this approach for calculation the matrices $\mathbf{M}^{1k}$ we use the matrix formalism for Lie algebraic methods. Following to these tools we use Lie maps representation for solution of the Eqs.(1)-(2):

$$X(t) = \mathcal{M}(t|t_0) \circ X_0 = \mathrm{T}\exp\left\{\mathcal{L}_F\right\} \circ X_0, \tag{4}$$

where $\mathcal{L}_F$ is a Lie operator associated with the function F ($\mathcal{L}_F = F^*(X_0,t)\partial/\partial X_0$), $\mathrm{T}\exp$ is the so called T-exponential operator (time ordered exponential operator). For non-autonomous cases this operator can be rewritten with the help of the Magnus's representation [6] in the form of routine exponential operator

$$\mathrm{T}\exp\left\{\mathcal{L}_F\right\} = \exp\left\{\hat{\mathcal{L}}(F;t|t_0)\right\}. \tag{5}$$

The new operator $\hat{\mathcal{L}}$ is associated with a new function $\hat{F}(F;t|t_0)$ which one can calculated using the continuous analogue of the CBH formula. The expansion (2) generates Taylor expansions of Lie operators $\mathcal{L}_F = \sum_{k\geq 0} \mathcal{L}_{F_k}$ and $\hat{\mathcal{L}}_F = \sum_{k\geq 0} \mathcal{L}_{\hat{F}_k}$. Here F_k and $\hat{F}_k$ are homogeneous vector polynomial functions, f.e. $F_k = \mathbf{P}_{1k} X^{[k]}$. The sequences of $\{F_k\}$ and $\left\{\hat{F}_k\right\}$ are defined by the following matrices $\mathbf{P}^{1k}(t)$ and $\hat{\mathbf{P}}^{1k}(t|t_0)$. Rewrite the Eqs.(4)-(5) using the analogue of the Dragt-Finn factorization in the form

$$\mathcal{M}(t|t_0) = \ldots \exp\left\{\tilde{\mathcal{L}}_3\right\} \circ \exp\left\{\tilde{\mathcal{L}}_2\right\} \circ \exp\left\{\tilde{\mathcal{L}}_1\right\}, \tag{6}$$

where $\tilde{\mathcal{L}}_k = \left(\mathbf{G}^{1k} X^{[k]}\right)^* \partial/\partial X$. For new matrices $\mathbf{G}^{1k}(F;t|t_0)$ were obtained the rather simple formulas using the Kronecker product and sum representations. Interacting by the factorized maps (6) on X_0 we evaluate the solution (3). The matrices $\mathbf{M}^{1k}$ can be written in closed forms as functions of $\mathbf{G}^{1k}$.

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2 THE EXPLICIT SOLUTIONS

2.1 The general features

It is known that the approach mentioned above is one of the perturbation theory methods. The desired solution is created in the form of power series (see the Eq.(1)). It is clear that way can be realized only with truncated procedures for some chosen order of expansions. In the referred works the corresponding matrices $\mathbf{P}^{1k}$, $\hat{\mathbf{P}}^{1k}$, $\mathbf{G}^{1k}$ and $\mathbf{M}^{1k}$ are calculated up to fifth order in symbolic forms using the computer algebra codes ($REDUCE$ for example). But we have to note that for this approach there appear two problems: *support of accuracy of truncated expansions* and *support of intrinsic properties (f.e. symplecticity for Hamiltonian systems).* The second problem can be solved with the help of the correction procedure [7] for the matrices $\mathbf{M}^{1k}$. For this correction we have to solve a chain of linear algebraic equations and redefined some of the elements of $\mathbf{M}^{1k}$. These calculations one can make in symbolic forms too. But for many applications (f.e. for long time tracking) it is very important to have solutions in explicit forms. These explicit forms can be used in two ways: for more accurate tracking of particles beams and for checking of approximate calculations.

2.2 The solution method

For the creation of explicit forms for Lie transformations $\mathcal{M}$ we use the matrix representation (3). At first we represent the designed solution $X(X_0;t|t_0)$ in the form

$$X(X_0;t|t_0) = \frac{W(X_0;t|t_0)}{v(X_0;t|t_0)}, \tag{7}$$

where $W(X_0;t|t_0)$ and $v(X_0;t|t_0)$ are analytical functions, $W = \sum_{k\geq 0} \mathbf{W}^{1k}(t|t_0)X_0^{[k]}$ and $v = \sum_{k=0}^{\infty} \left(V^{1k}(t|t_0)\right)^* X_0^{[k]}$, where $\mathbf{W}^{1k}$, V^{1k} are matrices and vectors of coefficients which must be calculated. Using the representation (3) and (7) we can write

$$\left(\sum_{l=0}^{\infty} \left(V^{1l}\right)^* X_0^{[l]}\right) \cdot \left(\sum_{k=0}^{\infty} \mathbf{M}^{1k} X_0^{[k]}\right) = \sum_{j=0}^{\infty} \mathbf{W}^{1j} X_0^{[j]}.$$

This expression can be rewritten using the Kronecker product properties in the form

$$\sum_{l=0}^{\infty} \left(\sum_{k=0}^{l} \left(V^{1k}\right)^* \otimes \mathbf{M}^{1\,l-k}\right) X_0^{[l]} = \sum_{j=0}^{\infty} \mathbf{W}^{1j} X_0^{[j]}. \tag{8}$$

In the Eq.(8) the matrices $\mathbf{M}^{1j}$ have to be defined or calculated from the initial motion equations (see the Eqs.(1)-(2)). Obviously that the system of linear algebraic equations (8) does not define unknown matrices and vectors $\mathbf{W}^{1k}$ and V^{1j} in a full measure. We have an arbitrariness which can be removed by superposition of additional conditions. As such conditions the symmetries conditions on the initial dynamic system can be suggested.

3 ORGANIZATION OF CALCULATIONS

3.1 The general case

Following to the approach described in the referred papers the calculation procedures is realized using some databases of corresponding matrices and formulas for abstract (noncommutative) variables [8]. The prepared formulas are used for necessary calculations for obtaining of explicit forms of solutions in the following ways: at first we impose some conditions on $\mathbf{P}^{1k}$ matrices included to the series in the Eq.(2). Then according to the matrix formalism we use necessary formulas for the solution matrices $\mathbf{M}^{1j}$ and solve the Eqs.(8). The block structure of matrices $\mathbf{P}^{1k}$ and $\mathbf{M}^{1j}$ correspondingly helps to do all calculations more comfortable.

3.2 Some simple examples

At first let give as an example the matrix representation for the Lie transformation associated with homogeneous polynomials $G_m = \mathbf{G}_m X^{[m]}$:

$$\exp\{\mathcal{L}_{G_m}\} \circ X =$$

$$= \sum_{k=0}^{\infty} \prod_{j=0}^{k} \frac{\mathbf{G}_m^{\oplus((j-1)(m-1)+1)}}{k!} X^{[k(m-1)+1]}. \tag{9}$$

Here $\mathbf{G}^{\oplus k}$ denotes the k-multiple Kronecker sum. In works [9, 10] some approach to the problems of explicit solutions for Hamiltonian systems is discussed. In this paper we suggest an alternative method which is usefulness for more general dynamics systems. Besides, we can use for solving corresponding equations computer algebra methods and codes. Also we can create some specialized data–bases which can help to generate necessary explicit solutions more flexible and effective. Consider a homogeneous polynomial of second order $G_2(X) = \mathbf{G}_2 X^{[2]}$ for simple case of $n = 2$, where

$$\mathbf{G}_2 = \begin{pmatrix} a & 0 & 0 \\ b & -2a & 0 \end{pmatrix},$$

where a and b are arbitrary constants. We propose that the desired solution has the following form $\sum_{k=0}^{m} \mathbf{W}^{1k} X^{[k]} / \sum_{l=0}^{n} \left(V^{1l}\right)^* X^{[l]}$. Solving the Eq.(7) we obtain the following solution matrices $\mathbf{W}^{1l}$ and vectors V^{1k} for $l = 0 \dots 4$, $k = 0 \dots 1$

$$\mathbf{W}^{10} = 0, \quad \mathbf{W}^{11} = \mathbf{E},$$

$$\mathbf{W}^{12} = \begin{pmatrix} 0 & 0 & 0 \\ -b & 3a & 0 \end{pmatrix}, \; \mathbf{W}^{13} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -b & 3a & 0 & 0 \end{pmatrix},$$

$$\mathbf{W}^{14} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ -b & 3a & 0 & 0 & 0 \end{pmatrix}, \tag{10}$$

$$V^{10} = 0, \quad V^{11} = a \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{11}$$

It is not difficult to see that the corresponding expression for $\exp\{\mathcal{L}_{G_2}\} \circ X$ (with regard to the Eqs. (3),(4),(7),(10) and (11)) is the same as expressions in [10] (up to notations) for a Hamiltonian system with a cubic term in the form $\mathcal{H}_3 = -bx^3/3 + ax^2p$, where x, p are components of the phase vector for $n = 2$. A similar expression (up to permutation of lines in the matrices $\mathbf{W}^{1k}$ and vectors V^{1l} and changing the constant a to a new constant c) can be obtained for a polynomial $\mathcal{H}_3 = -axp^2 + cp^3/3$.

In conclusion I would like to point some moments. In the first place, the suggested method is based on the unique mathematical tools which are used in the frame of the matrix formalism. Secondly, this approach allows data-bases of necessary matrices to be created for advance. And finally, it is possible to use computer algebra methods and codes for all necessary calculations. Ultimately this provides a possibility of creation of a prototype of expert systems for particle beam lines modelling and optimization.

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