EVALUATION OF DYNAMIC APERTURE IN THE PRESENCE OF PHASE SPACE DISTORTIONS*

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Abstract

We study symplectic mappings which occur in the modelization of the 4D betatronic motion in a magnetic lattice and define the dynamic aperture in terms of the connected volume in phase space of initial conditions which are bounded for a given number of iterations. Different methods for a fast estimate of the dynamic aperture are outlined; the analysis of the associated errors and the optimization of the integration steps are also reviewed. The accuracy of the different approaches have been tested by mean of numerical simulations. Both simple models and more realistic lattices have been considered.

1 INTRODUCTION

The presence of nonlinearities in the magnetic field of the elements of an accelerator can greatly reduce the stability domain [1, 2, 3]. An accurate estimate of the dimension of this domain is crucial both for the understanding of the dynamics of existing machines [3] and for the specification of the lattice parameters of planned machines [4].

The numerical estimate of the dynamic aperture is related to the computation of the volume in phase space of the initial conditions that are stable after a given number of revolutions around the machine. The numerical evaluation of this volume is very CPU time consuming, as in principle one should scan the four variables (x, p_x, y, p_y).

To overcome these problems for complicated lattices, the tracking is carried out over initial conditions with $p_x = p_y = 0$ and a fixed ratio x/y with a large gain in the CPU time [2, 3]. Unfortunately, this approach does not take into account two main effects, i.e. the distortion of the orbits along the phases [5] and the different dynamics of the particles with various ratios x/y [1, 3, 6]. Neglecting these effects, the computed dynamic aperture will be affected by errors that cannot be estimated a priori.

We present here some original numerical methods [7]: to evaluate the dynamic aperture taking into account the phase space distortions. We prove that it is possible to exploit the dynamics to take into account the distortion of the orbits along the phases, thus avoiding the integration over these variables. We develop two algorithms to carry out these fast estimates: one is based on numerical integration [8], the second exploits the perturbative tools of normal forms [9, 10, 11].

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2 DYNAMIC APERTURE DEFINITION

Let $\mathbf{x} = (x, p_x, y, p_y)$ be the vector of the Courant-Snyder coordinates at a given section of the machine. The linear motion is the direct product of two constant rotations in the planes (x, p_x) and (y, p_y) by the linear tunes. Let us consider the phase space volume of the initial conditions that are bounded after N iterations:

$$\int\int\int\int \chi(x, p_x, y, p_y)\; dx\; dp_x\; dy\; dp_y, \qquad (1)$$

where $\chi(x, p_x, y, p_y)$ is the characteristic function of the set of initial conditions that are bounded under N iterations. Since in 4D the invariant curves (i.e. 2D KAM tori) do not separate different domains of phase space, there does not exist a last invariant curve surrounding stable initial conditions [12, 11]. However, it seems from numerical simulations [2, 3, 6, 13, 14] that pathological situations are not typical of weakly nonlinear lattices and they have no practical relevance, since they occupy a negligible fraction of the phase space volume. Therefore, in general, there exists a connected region of initial conditions that are stable for a given number of iterations.

3 METHODS TO COMPUTE THE 4D DYNAMIC APERTURE

3.1 Method 1: direct integration.

To exclude the disconnected part of the stability domain in the integral (1), we have to choose a suitable coordinate transformation. The natural choice is to use polar variables $(r_1, \vartheta_1, r_2, \vartheta_2)$; r_1 and r_2 are the linear invariants. As the nonlinear part of the equations of motion adds a coupling between the two planes, it is natural to replace r_1 and r_2 with the polar variables $r\cos\alpha$ and $r\sin\alpha$:

$$\begin{cases} x &= r\cos\alpha\cos\vartheta_1 \\ p_x &= r\cos\alpha\sin\vartheta_1 \qquad\qquad r\in[0,+\infty[\\ y &= r\sin\alpha\cos\vartheta_2 \qquad\qquad \vartheta_1,\vartheta_2\in[0,2\pi[\\ p_y &= r\sin\alpha\sin\vartheta_2 \qquad\qquad \alpha\in[0,\pi/2]; \end{cases}$$

Having fixed α, ϑ_1 and ϑ_2, let $r(\alpha, \vartheta_1, \vartheta_2)$ be the first value of r whose orbit is not bounded after N iterations. Then, we define the dynamic aperture as the radius $r_{\alpha,\vartheta_1,\vartheta_2}$ of the hypersphere that has the same volume as the stability domain. To evaluate numerically this quantity, one considers J steps in the radial variable, K steps in the angle α and L steps in the angles ϑ_1, ϑ_2: the dynamic aperture then reads [7].

$$r^4_{\alpha,\vartheta_1,\vartheta_2} = \frac{\pi}{2KL^2} \sum_{k=1}^{K} \sum_{l_1,l_2=1}^{L} [r(\alpha_k, \vartheta_{1l_1}, \vartheta_{2l_2})]^4 \sin(2\alpha_k)$$

Error sources. The discretization both in the angular and in the radial variables leads to an integration error, which can be estimated using the standard tools of numerical analysis.

- The discretization in the angles ϑ_1 and ϑ_2 corresponds to a trapezoidal rule of integration [15]. In the followings we will always assume that the more pessimistic estimate L^{-1} holds.
- The discretization in the angle α gives a relative error proportional to K^{-1}.
- The discretization in the radius r gives a relative error proportional to J^{-1}.

Step optimization. One should choose integration steps that produce comparable errors, i.e. $J \propto K \propto L$. In this way, neglecting the constant factors, one can obtain a relative error of $1/(4J)$ by evaluating J^4 orbits, i.e. NJ^4 iterates. The fourth power comes from the dimensionality of phase space, and makes a precise estimate of the dynamic aperture very CPU time consuming.

3.2 Method 2: integration over the dynamics.

The direct integration method, contains the average of $r(\alpha, \vartheta_1, \vartheta_2)^4$ over the angles. It is possible to replace such an average with an average over the iterates. To avoid the effects of the non-uniformity of the distribution of the phases on the last invariant curve, one can proceed in the following way [8]

- We fix $\overline{\vartheta}_1$ and $\overline{\vartheta}_2$. A scan over α is performed to find the radius $r(\alpha, \overline{\vartheta}_1, \overline{\vartheta}_2)$ defined in the previous section and at the same time the N iterates of the orbit are computed.
- The square $[0, 2\pi[\times[0, 2\pi[$ is divided in M^2 equal squares (with $M^2 \leq N$), such that each square contains at least the phase of one iterate of the last stable curve.
- For each square (m_1, m_2), where $m_1 = 1, ..., M$ and $m_2 = 1, ..., M$, we compute $r_{m_1,m_2}(\alpha, \overline{\vartheta}_1, \overline{\vartheta}_2)$, that is the average distance to the origin of the iterates that fall in that angular square.

Finally, the dynamic aperture is computed as

$$r^4_{\alpha,d} = \frac{\pi}{2KM^2} \sum_{k=1}^{K} \sum_{m_1,m_2=1}^{M} [r_{m_1,m_2}(\alpha_k, \overline{\vartheta}_1, \overline{\vartheta}_2)]^4 \sin(2\alpha_k).$$

Error sources. The error is given by the following contributions.

- The discretization in the angles ϑ_1, ϑ_2, which is given by the M^2 squares over which the integration is carried out. The relative error in the dynamic aperture is proportional to $M^{-1} \propto N^{-1/2}$.
- Discretization in the angle α: the relative error is proportional to K^{-1}.
- Discretization in the radius r: the relative error is proportional to J^{-1}.

Step optimization. One should choose $J \propto K \propto \sqrt{M}$. Neglecting the multiplicative constants, one obtains a relative error of $1/(4J)$ evaluating J^2 orbits, i.e. $J^2M^2 \propto J^2N$ iterates, thus saving a factor J^2 with respect to Method 1.

3.3 Method 3: normal forms.

According to the nonresonant normal form theory, using a conjugating function $\mathbf{\Phi}$ one transforms a 4D map $\mathbf{F}$ into its normal form $\mathbf{U}$ [10, 11], namely a direct product of rotations in the two phase planes (x, p_x) and (y, p_y), whose nonlinear frequencies depend on the distance to the origin. The two components of the inverse conjugating function Ψ_1 and Ψ_2 give the approximated nonlinear invariants ρ_1 and ρ_2. Thanks to the properties of the normal forms, the nonlinear invariants ρ_1, ρ_2 will be independent on the values of $\overline{\vartheta}_1, \overline{\vartheta}_2$ and the integration over the phases can be trivially computed. The first order result will be

$$r^4_{\alpha,nf} = \frac{\pi}{2K} \sum_{k=1}^{K} [\rho_{1,k}+\rho_{2,k}]^2 \sin(2\alpha_k) \;\; \rho_{i,k} = \rho_i(\alpha_k, \overline{\vartheta}_1, \overline{\vartheta}_2).$$

Error sources. The error is given by the following contributions.

- Discretization in the angle α: the relative error is proportional to K^{-1}.
- Discretization in the radius r: the relative error is proportional to J^{-1}.
- Normal form error. The application of normal forms close to the dynamic aperture can give inaccurate results [11]. This error is due to the divergence of the perturbative series and to the truncation of the series. In the numerical examples analyzed in this paper, the linear frequencies are far from low order resonances and the normal forms turn out to be very accurate.

Step optimization. One should choose $J \propto K$. Neglecting the multiplicative constants and assuming that the normal form error is smaller than the integration error over r and α, one obtains a relative error of $1/(4J)$ by evaluating J^2 orbits, i.e. J^2N iterates: one saves a factor J^2 with respect to Method 1 (without constraints over the number of iterates such as in Method 2).

Table 1: Dynamic aperture estimates for LHC and SPS.

Model	Average Relative Error w.r.t. $r_{\alpha,\theta_1,\theta_2}$		
	r_0	$r_{\alpha,d}$	$r_{\alpha,nf}$
LHC - Sex. only	16%	2%	3%
LHC - All mult.	9%	1.5%	2%
SPS - WP1	13%	9%	8%
SPS - WP2	37%	5%	6%

4 NUMERICAL RESULTS

LHC cell lattice with random errors We consider a lattice made up of 8 LHC–like cells [4] plus a phase shifter to set the linear tunes to the values $\nu_x = 0.28$, $\nu_y = 0.31$. Two different sets of nonlinearities have been considered: a lattice with only random sextupolar components in the dipoles, and a lattice with random sextupolar, octupolar and decapolar components in the dipoles. The estimated values of the LHC dipole errors have been used. For each case we analysed 10 different seeds. In Tab. 1 we report the relative errors between Methods 2, 3, and Method 1. We also give the position r_0 of the last invariant curve along the direction $\alpha = \pi/4$ and $\vartheta_1 = \vartheta_2 = 0$; this indicator is commonly used for fast dynamic aperture estimates of complicated lattices [2, 3].
We computed the dynamic aperture over $N = 1000$ turns using 20 steps for each variable, giving an accuracy of 2%. For the Methods 2, 3 the number of steps in α and in r is 20; $r_{\alpha,d}$ is computed over 1000 iterates. The normal form truncation is fixed between 3 and 8, choosing the order that minimizes the normal form error. The results show that both $r_{\alpha,d}$ and $r_{\alpha,nf}$ provide an estimate of the dynamic aperture which is in agreement with the direct integration of the stability domain, without scanning over the two angles ϑ_1 and ϑ_2. The estimate r_0 neglects both the distortion of the orbit and the contributions coming from particles with different emittances: as these phenomena are relevant, this estimate is rather imprecise.
SPS lattice We also consider the SPS lattice corresponding to the set–up used for nonlinear dynamics experiments [16]. The nonlinear part of the lattice consists of 8 strong extraction sextupoles plus 108 chromatic sextupoles. Two working points have been considered: the first one (WP1) at $\nu_x = 26.637$ and $\nu_y = 26.533$, which is close to resonances of order 7 and 8; the second one (WP2) is $\nu_x = 26.605$ and $\nu_y = 26.538$, which is close to resonances of order 5. Both cases correspond to very perturbed situations where the nonlinear resonances are excited and the phase space is strongly deformed.
In Tab. 1 the different estimates of the dynamic aperture r_0, $r_{\alpha,d}$ and $r_{\alpha,nf}$ are compared to the estimate $r_{\alpha,\vartheta_1,\vartheta_2}$ computed with 20 steps in each variable. The results show that, due to the high distortion in phase space, the estimate r_0 is really imprecise. On the other hand, Methods 2 and 3 provide a better estimate, even if the errors are considerably higher than in the other cases; this is probably due to the strong nonlinearities of these models.

5 CONCLUDING REMARKS

In this paper we have presented three methods to compute the dynamic aperture and to estimate the associated errors. The optimization of the integration steps have been discussed as well.
Method 2 and 3 have given good results showing that the dependence on the phases and on the ratio of emittances can be crucial for obtaining a precise estimate of the dynamic aperture for realistic models.
As these numerical results are strongly model–dependent, we believe that for each model one should carefully test the relevance of these effects to choose the best compromise between accuracy and CPU time.

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