Orbit dynamics in a low energy electron linear accelerator.

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Abstract

A relativistic Hamiltonian is taken as starting point for analytical calculations of the particle motion in a low energy electron linear accelerator. In this Hamiltonian the electromagnetic space waves are represented as vector potentials. Also a longitudinal focusing magnetic field is incorporated by means of its vector potential. For an adequate representation of the latter, one has to use cartesian coordinates. After a few canonical transformations a final Hamiltonian is obtained that gives insight in several physical phenomena related to the particle motion. The effect of the energy increase along the accelerator on the transverse focusing is taken into account. The results are correct up to second order in the amplitudes of the waves.

1 INTRODUCTION

The orbit dynamics in electron linear accelerators are rather complicated due to a series of electromagnetic (EM) waves with different propagation constants. Aside the accelerating wave the harmonic spatial waves following Floquet's theorem, do occur. Further there is a large increase of energy per period. Several focusing effects are generally treated separately, e.g. phase focusing, ponderomotive focusing and magnetic focusing [1] [2] [3] [4]. Many of these effects are rather similar to those present in A.V.F. cyclotrons and A.G. synchrotrons, for which analytical theories using Hamilton mechanics yield reliable results [5].

Often the description of the particle motion is given in cylindrical coordinates resulting in a problem for the description of particles in linacs with non zero azimuthal momentum.

In this paper the EM fields are represented by their magnetic vector potentials, the applied coordinate system is cartesian with the z-axis pointing along the optical axis. After a number of successive canonical transformations a Hamiltonian is constructed with slowly varying coefficients. Then an adiabatical treatment can be carried out. This contribution deals with linear motion. However, the theory is sufficiently general to be further developed for non linear motion, including coupling effects.

2 THE ELECTROMAGNETIC VECTOR POTENTIAL

The components of the vector potential representing the EM waves in polar coordinates are given by

$$\begin{aligned} A_z &= \tfrac{-1}{\omega} E_n J_0(\alpha_n r)\sin(k_n z - \omega t) \\ A_r &= \tfrac{k_n}{\omega} E_n \tfrac{1}{\alpha_n} J_1(\alpha_n r)\cos(k_n z - \omega t) \end{aligned} \tag{1}$$

Here J_0 and J_1 are modified Bessel functions, α_n is the n-th zero of J_0, E_n is the electric field amplitude of the n-th mode, $k_n = k_f + \frac{2\pi n}{d}$, $\alpha_n^2 = k_n^2 - k^2$, k is the propagation constant in vacuum, k_f is the propagation constant of the accelerating EM wave which is taken constant. For the linear theory $J_0 = 1 + \frac{1}{4}\alpha_n^2 r^2$, $J_1 = \frac{1}{2}\alpha_n r$. In cartesian coordinates the representation is

$$\begin{aligned} A_z &= \tfrac{-1}{\omega} E_n J_0(\alpha_n r)\sin(k_n z - \omega t) \\ A_x &= \tfrac{k_n}{2\omega} E_n x\cos(k_n z - \omega t) - \tfrac{1}{2}B(z)y \\ A_y &= \tfrac{k_n}{2\omega} E_n y\cos(k_n z - \omega t) + \tfrac{1}{2}B(z)x, \end{aligned} \tag{2}$$

where $B(z)$ represents an added z-dependent solenoidal magnetic field.

3 THE BASIC HAMILTONIAN

The relativistic motion is described by the Hamiltonian

$$\begin{aligned} H = \; & \left\{E_R^2 + (p_x - eA_x)^2 c^2 \right. \\ & \left. +(p_y - eA_y)^2 c^2 + (p_z - eA_z)^2 c^2\right\}^{1/2}, \end{aligned} \tag{3}$$

with p_x, p_y the transversal and p_z the longitudinal components of the canonical momentum, E_R the rest mass energy and c the speed of light. It is convenient to switch over to a new Hamiltonian $K = -p_z$. Then the z-coordinate is the new independent variable, $p = -H$ and t form a pair of new conjugated canonical variables. By this manipulation trajectories, particle energy and phase are described as a function of position along the optical axis. Relative momenta, relative field quantities and a new coordinate $\zeta = ct$ are introduced:

$$\begin{aligned} &\pi_x = \tfrac{p_x}{W_0/c} \quad , \quad \pi_y = \tfrac{p_x}{W_0/c} \quad , \quad h = \tfrac{-H}{W_0} \\ &\mathcal{E}_n = \tfrac{eE_n}{W_0 k} \quad , \quad b = \tfrac{eB(z)c}{W_0 k} \quad , \quad e = \tfrac{E_R}{W_0}, \end{aligned} \tag{4}$$

with W_0 a constant reference energy. A transformation of ζ is carried out by the generating function G_0:

$$G_0(\pi_x, \overline{x}, \pi_y, \overline{y}, h, \overline{\zeta}, z) = -h\overline{\zeta} - zh\frac{c}{v_f} - \pi_x\overline{x} - \pi_y\overline{y}$$
$$\zeta = -\frac{\partial G_0}{\partial h} = \overline{\zeta} + \frac{zc}{v_f}$$
$$\overline{h} = -\frac{\partial G_0}{\partial \overline{\zeta}} = h$$
$$\overline{K} = K + \frac{\partial G_0}{\partial z} = K - h\frac{c}{v_f} = K - h\frac{k_f}{k}. \tag{5}$$

After this transformation the Hamiltonian becomes

$$\begin{aligned} \overline{K} \;=\; & -\{h^2 - e^2 - \\ & (\pi_x - \tfrac{1}{2}\mathcal{E}_n k_n x\cos(\tfrac{2\pi n}{d}z - \tfrac{\omega\overline{\zeta}}{c}) + \tfrac{1}{2}bky)^2 + \\ & (\pi_y - \tfrac{1}{2}\mathcal{E}_n k_n y\cos(\tfrac{2\pi n}{d}z - \tfrac{\omega\overline{\zeta}}{c}) - \tfrac{1}{2}bkx)^2\}^{1/2} \\ & - \mathcal{E}_n J_0(\alpha_n r)\sin(\tfrac{2\pi n}{d}z - \tfrac{\omega\overline{\zeta}}{c}) - h\tfrac{k_f}{k}. \end{aligned} \tag{6}$$

This Hamiltonian looks rather complex, but after two successive transformations it becomes a clear presentation.

4 PHASE FOCUSING AND MAGNETIC FOCUSING

Two successive transformations are applied. The first one transforms the canonical momenta π_x and π_y into new ones that equal the kinetic momenta. The second one introduces a transformation to a coordinate system rotating around the z-axis with half the cyclotron frequency.

Transformation I

$$G_1(\pi_x, \overline{x}, \pi_y, \overline{y}, h, \overline{\zeta}, z) = -h\overline{\zeta} - \pi_x\overline{x} - \pi_y\overline{y}$$
$$+\tfrac{1}{4}\mathcal{E}_n k_n \overline{x}^2\cos(\tfrac{2\pi n}{d}z - \tfrac{\omega\overline{\zeta_0}}{c})$$
$$+\tfrac{1}{4}\mathcal{E}_n k_n \overline{y}^2\cos(\tfrac{2\pi n}{d}z - \tfrac{\omega\overline{\zeta_0}}{c}), \tag{7}$$
$$\overline{\pi}_x = -\frac{\partial G_1}{\partial \overline{x}} = \pi_x - \tfrac{1}{2}\mathcal{E}_n k_n \overline{x}\cos(\tfrac{2\pi n}{d}z - \tfrac{\omega\overline{\zeta_0}}{c})$$
$$x = -\frac{\partial G}{\partial \pi_x} = \overline{x}, \quad \text{similar eqs. for } \overline{\pi}_y \text{ and } y.$$
$$\overline{K} = K + \frac{\partial G_1}{\partial z}.$$

The phase $\frac{\omega}{c}\overline{\zeta}$ may be seen as a small quantity giving rise to coupling in non linear terms. This is taken as a constant, $\frac{\omega}{c}\overline{\zeta_0}$, in the derivation of the transversal part of the Hamiltonian. One observes that the vector potential is removed from the canonical momenta π_x, π_y. It returns as a potential-like function via $\frac{\partial G_1}{\partial z}$.

Transformation II This transformation is applied after expanding the square root in eq. 6.

$$G_2(\overline{\pi}_x, x, \overline{\pi}_y, y, \overline{h}, \zeta) = \overline{h}\zeta$$
$$+\overline{\pi}_x x\cos\phi + \overline{\pi}_x y\sin\phi - \overline{\pi}_y x\sin\phi + \overline{\pi}_y y\cos\phi,$$
$$\overline{x} = \frac{\partial G_2}{\partial \overline{\pi}_x}$$
$$\pi_x = \frac{\partial G_2}{\partial x}$$
$$\overline{K} = K + \frac{\partial G_2}{\partial z}$$
$$\text{and} \quad \phi = -\int \tfrac{1}{2}\frac{bck}{v_p}dz. \tag{8}$$

The resulting Hamiltonian for only the main wave $\mathcal{E}_0$ becomes

$$\begin{aligned} K_2 \;=\; & -(h^2 - e^2)^{1/2} - \mathcal{E}_0\sin(\tfrac{\omega\overline{\zeta}_0}{c}) - h\tfrac{k_f}{k} \\ & +\tfrac{1}{2}\frac{\pi_x^2 + \pi_y^2}{\sqrt{h_0^2 - e^2}} \\ & -\tfrac{1}{2}x^2\{\tfrac{1}{2}\mathcal{E}_0(k_f k_p - k^2)\sin(\tfrac{\omega\overline{\zeta}_0}{c}) - \tfrac{1}{4}\frac{b^2k^2}{\sqrt{h_0^2 - e^2}}\} \\ & -\tfrac{1}{2}y^2\{\tfrac{1}{2}\mathcal{E}_0(k_f k_p - k^2)\sin(\tfrac{\omega\overline{\zeta}_0}{c}) - \tfrac{1}{4}\frac{b^2k^2}{\sqrt{h_0^2 - e^2}}\}, \end{aligned} \tag{9}$$

the subscript p refers to the particle. After a transformation has been carried out bars are removed. This is only convenient if the meaning of a variable is not changed too much. In the transversal part of the Hamiltonian the synchronous solution of the longitudinal motion is substituted (h_0, $\overline{\zeta}_0$). Taking the synchronous phase as a small quantity of first order it appears that the complete focusing in eq. 9 is of second order: $\mathcal{E}_0\sin(\omega\overline{\zeta_0}/c)$, b^2. In fact the phase focusing becomes much smaller towards higher energies.

5 FOCUSING DUE TO OSCILLATING TERMS

Rapidly oscillating terms of first order yield second order “constant“ terms after their removal by a suitable canonical transformation, giving the ponderomotive action. These terms occur in the longitudinal solution for the synchronous particle, in the potential function generated by G_1 and are due to the transformation from H to $\overline{K}$.

One has

$$\begin{aligned} \Delta K_2 &= \tfrac{1}{2}(x^2 + y^2)\mathcal{E}_n\tfrac{\pi n k}{d}\sin(\tfrac{2\pi n}{d}z), \\ h_0 &= \mathcal{E}_0\tfrac{2\pi}{\lambda}z + \tfrac{d}{n\lambda}\mathcal{E}_n\sin(\tfrac{2\pi n}{d}z) - w_i, \end{aligned} \tag{10}$$

w_i represents the initial energy, its sign arises from the fact that the momentum h has a negative sign with respect to H. The first term on the right-hand side of the second equation represents the steady increase of energy.

All first order oscillating terms can be transformed away by a third transformation

$$
\begin{aligned}
& G_3(\pi_x, \overline{x}, \pi_y, \overline{y}, h, \overline{\zeta}) = -h\overline{\zeta} - a\pi_x\overline{x} - a\pi_y\overline{y} + b(\overline{x}^2 + \overline{y}^2) \\
& a^2 = \frac{1}{\sqrt{h_0^2 - e^2}} \quad , \quad b = \frac{1}{2a}\frac{da}{dz} \\
& \overline{\pi}_x = -\frac{\partial G_3}{\partial \overline{x}} \\
& x = -\frac{\partial G_3}{\partial \pi_x},
\end{aligned}
\tag{11}
$$

The final Hamiltonian becomes

$$
\begin{aligned}
K_f \quad &= (h^2 - e^2)^{1/2} - \mathcal{E}_0 \sin(\tfrac{\omega\overline{\zeta}}{c}) - h\tfrac{k_f}{k} \\
&+ \tfrac{1}{2}\pi_x^2 + \tfrac{1}{2}\pi_y^2 \\
&+ \tfrac{1}{2}(x^2 + y^2)\{-\tfrac{1}{2}\tfrac{\mathcal{E}_0}{\sqrt{h_0^2 - e^2}}(k_f k_p - k^2)\sin(\tfrac{\omega\overline{\zeta_0}}{c}) \\
&+ \tfrac{1}{4}\tfrac{b^2k^2}{h_0^2 - e^2} + \tfrac{k^2}{h_0^2}\left(\tfrac{1}{4}\mathcal{E}_0^2 + \tfrac{1}{8}(\mathcal{E}_n + \mathcal{E}_{-n})^2\right)\},
\end{aligned}
\tag{12}
$$

where for simplicity in the derivation of the last focusing term $\frac{h_0^2}{h_0^2 - e^2}$ is approximated by unity. The oscillating part in h_0 now has been removed. All field quantities in the focusing terms are now roughly divided by the energy $|h_0|$. The variation of the focusing term is slow and can be treated adiabatically. In standing wave structures, e.g. a Π-mode structure, $\mathcal{E}_{-1}$ equals $\mathcal{E}_0$. For other modes one has to repeat carefully the transformation G_3.

6 REMARKS

The final Hamiltonian K_f is a good representation from which the particle motion can be derived. The coefficients for the transversal motion are accurate up to second order in the wave amplitudes. However, in deriving the ponderomotive action the approximation $h_0^2 >> e^2$ has been used. Taking the constants k_f and k_p equal one gets $k_f^2 - k^2 = \frac{e^2}{h_0^2 - e^2}$. For high energies this quantity can be seen as a quantity of at least order one. Therefore the phase focusing becomes of minor importance compared to both other terms. For low energies compared to the rest mass energy $(h_0^2 - e^2)/e^2$ may be taken as a quantity of order one. Then the phase focusing and the magnetic focusing become important. The transformation G_3 can be performed again showing the first order phase focusing and the second order ponderomotive focusing that occur in a cavity when the energy increase is small compared to the kinetic energy of a particle. For $(h_0^2 - e^2)/e^2 \simeq 1$ the transformation G_3 becomes more complex and must be followed by a fourth transformation.

If the wave propagation factor is depending on the axial position z, the arguments of the $\cos / \sin$-functions must be changed into $(\int k(z)dz - \omega t)$. Solving the wave equation shows that at the same time the amplitude of the electric field strength changes with $z : E \sim 1/\sqrt{k(z)}$. In the Hamiltonian of eq. 6 an extra term $x\frac{\partial E_n}{\partial z}\sin(\ldots)$ will appear. Then after transformation G_1 extra potential terms arise already for the main wave that may be of the same order as the phase focusing if $\frac{d}{E_0}\frac{\partial E_0}{\partial z}$ is not sufficiently small. In an electron linear accelerator this will happen in the first few cells of the waveguide. It is speculative to modify all Floquet waves into $E_n(z)J_0(\alpha_n(z)r)\sin(\int(k_0(z) + \frac{2\pi n}{d(z)})dz - \omega t)$, when $d(z)$ is taken either as smoothly varying or as discontinuously varying.

The Floquet waves are in fact a bad expansion to represent the EM fields. They do not converge towards the boundary conditions. A treatment that is given by Pruiksma et.al. [6] may then become more convenient. However, it remains complicated.

The ponderomotive focusing forces arise in fact from the alternating focusing and defocusing actions, where at the same time the energy of the particle differs from one action to the other.

Also for the longitudinal motion a second order focusing occurs due to the terms $\mathcal{E}_n \sin(\frac{2\pi n}{d} - \frac{\omega\overline{\zeta}}{c})$ as given in eq. 6. They arise when a particle with a phase deviation e.g. first is accelerated somewhat more than a central particle and then decelerated. These actions each are in first order proportional to the phase deviation and will show at least a second order effect. A transformation like G_3 after first solving for a central particle has to be applied.

7 REFERENCES

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