HIGHER ORDER HARD EDGE END FIELD EFFECTS*

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Abstract

In most cases, nonlinearities from magnets must be properly included in tracking and analysis to properly compute quantities of interest, in particular chromatic properties and dynamic aperture. One source of nonlinearities in magnets that is often important and cannot be avoided is the nonlinearity arising at the end of a magnet due to the longitudinal variation of the field at the end of the magnet. Part of this effect is independent of the longitudinal of the end. It is lowest order in the body field of the magnet, and is the result of taking a limit as the length over which the field at the end varies approaches zero. This is referred to as a "hard edge" end field. This effect has been computed previously to lowest order in the transverse variables. This paper describes a method to compute this effect to arbitrary order in the transverse variables, under certain constraints.

INTRODUCTION

This paper computes the effect of the magnet end fields to first order in the magnitude of the magnetic field in the body of the magnet. Thus, at all points in this computation any effect which is of higher than first order in the magnitude of the magnetic field will be dropped. In addition, the fields are assumed to be varying only over a short distance. The computation will be done in the limit that this distance goes to zero. This effect will be shown to be independent of the longitudinal of the end field.

Computing this "hard-edge" end effect can be an important design tool. Computing the end field profile for a real magnet is very time-consuming. But a lattice design process must progress rapidly, and cannot re-design magnets every time the lattice parameters change. The hard-edge end effect allows one to have a reasonable estimate for the effects of the ends without knowing the details of the magnet construction. One can thus compute chromatic effects on the linear functions and dynamics apertures, for instance, that for some machines may be significantly affected by these end fields.

Performing the computation to lowest order in the body field should become more accurate as the magnet gets longer compared to its aperture. Most accelerator systems are designed such that the effect of the magnet ends is small compared to the effect of the body of the magnet. Furthermore, this computation finds an effect which is independent of the longitudinal profile of the end field; terms higher order in the end field are expected to depend on the longitudinal of the end field: imagine a kick-drift-kick combination where the kicks are taken at two different points in the end field. This effect is second order in the body magnetic field, but seems to depend on the precise longitudinal of the end.

The magnetic field as a function of distance s along a reference orbit is assumed to be proportional to a function of the transverse position times a function $\mathcal{S}_L(s)$. $\mathcal{S}_L(s)$ is zero for $s < -L/2$, one for $s > L/2$, and is infinitely differentiable everywhere. An example of such a function would be

$$\mathcal{S}_L(s) = \begin{cases} 0 & s < -L/2 \\ \dfrac{1}{1 + e^{-4\sqrt{3}sL/(L^2-4s^2)}} & -L/2 < s < L/2 \\ 1 & s > L/2. \end{cases} \tag{1}$$

If the field does not go from zero to a finite value, but instead goes between two constant values, the derivation will not change, and all that will matter is the change in the field from beginning to end.

Because all computations will involve integrating from $-L/2$ to $L/2$ and then taking the limit as $L \to 0$, the integral of any function of S_L will be zero in the limit $L \to 0$. Furthermore, it will be assumed that no integrals involve products of $\mathcal{S}_L$ with itself or its derivatives. This is equivalent to the statement that the computation will only be performed to first order in the field values, plus the assumption that the metric of the coordinate system (either for the field definition or the reference orbit) is not varying on the scale of L. This latter constraint requires special handling for magnets which are considered to be in a curvilinear coordinate system within the magnet and a straight coordinate system just outside the magnet. The correct handling of this situation must reflect the magnet construction: is the end of the magnet better represented as being straight or by curving with the body of the magnet?

This problem has been addressed to lowest nontrivial order in the transverse variables [1]. Here we show how to perform the computation to arbitrary order in the transverse variables.

LIE ALGEBRAIC COMPUTATION

Begin with the Hamiltonian in the form

$$H = H_p - H_q \tag{2}$$

H_p is independent of magnetic field, and H_q is first order in the magnetic field. Terms that are higher order in the

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magnetic field are dropped in this computation. We wish to compute the map going through the magnet end.

The map can be written in Lie algebraic notation as

$$e^{:f_p(s):}e^{:f_q(s):}, \tag{3}$$

where

$$\frac{d}{ds}e^{:f_p(s):} = -e^{:f_p(s):}{:}H_p{:}. \tag{4}$$

Only terms in f_q which are first order in H_q will be computed. In the limit $L \to 0$, $f_p(L/2) \to 0$, since H_p is finite.

One can write down a differential equation for f_q [2, 3]:

$$\operatorname{iex}(-{:}f_q{:})\frac{df_q}{ds} = H_q + (e^{-:f_q:} - 1)H_p, \tag{5}$$

where

$$\operatorname{iex}(x) = \frac{e^x - 1}{x} = 1 + \frac{x}{2} + \frac{x^2}{6} + \cdots. \tag{6}$$

Write f as a series

$$f_q(s) = \sum_{k=1} f_k(s). \tag{7}$$

Begin with

$$f_1(s) = \int_{-L/2}^{s} H_q(\bar{s})\, d\bar{s}. \tag{8}$$

Then, ignoring terms that are more than first order in H_q,

$$f_{n+1}(s) = \int_{-L/2}^{s} [H_p, f_n(\bar{s})]\, d\bar{s}. \tag{9}$$

The series does in fact converge, in the sense that each term is of higher order in the transverse variables than the next. First, note that in the limit $L \to 0$,

$$\int_{-L/2}^{L/2} ds_1 \int_{-L/2}^{s_1} ds_2 \cdots \int_{-L/2}^{s_{n-1}} ds_n\, \mathcal{S}_L^{(k)}(s_n) = \delta_{kn}. \tag{10}$$

Next, note that the term a magnetic field expansion satisfying Maxwell's equations that is proportional to the kth longitudinal derivative of the magnetic field has a higher minimum order in some quantities (usually the transverse coordinates) than the term proportional to the $(k-1)$st longitudinal derivative. Furthermore, to lowest order, H_p is second order in the transverse phase space variables. The result is that f_n is of higher order in the transverse variables than f_{n-1}.

Evaluating the map only need be done to first order in f_q, since the map is only correct to that order anyhow. One method which should work well is the implicit midpoint rule

$$\boldsymbol{z}_f = \boldsymbol{z}_i + f_q\left(\frac{\boldsymbol{z}_i + \boldsymbol{z}_f}{2}\right). \tag{11}$$

This happens to be second order in the transverse variables, but is probably not any slower than any first order method. Operator splitting methods are unlikely to work well, since f cannot easily be written as a sum of integrable pieces. One could conceive of writing it as a series of monomials, which are integrable [4, 5], but the implicit midpoint method is simpler and likely to be comparably fast to evaluate.

Accelerator Hamiltonian

To first order in the fields, the accelerator Hamiltonian in unscaled variables is

$$\begin{aligned} &-(1 + h_x x + h_y y)p_s - q(1 + h_x x + h_y y)A_s \\ &\qquad - q(1 + h_x x + h_y y)\frac{p_x A_x + p_y A_y}{p_s}, \end{aligned} \tag{12}$$

where

$$p_s = \sqrt{(E/c)^2 - (mc)^2 - p_x^2 - p_y^2}. \tag{13}$$

The first term above is H_p, and the sum of the last two terms is $-H_q$. Thus, $[H_p, f]$ is

$$\begin{aligned} &-\left[h_x p_s \frac{\partial f}{\partial p_x} + h_y p_s \frac{\partial f}{\partial p_y}\right. \\ &\qquad \left. + (1 + h_x x + h_y y)\left(\frac{p_x}{p_s}\frac{\partial f}{\partial x} + \frac{p_y}{p_s}\frac{\partial f}{\partial y}\right)\right]. \end{aligned} \tag{14}$$

EXAMPLE

Assume that $B_{y0}(x) = B_y(x, 0)$ is given in the body of the magnet; its change at the end of the magnet is $\Delta B_{y0}(x)$. Its variation is assumed to be a function of s times $B_{y0}(x)$. Maxwell's equations will give the components of the field which are higher order in y. Then the generating function f_q is, to the lowest three nontrivial orders in the vertical phase space variables,

$$
\begin{aligned}
f_q = {} & \frac{qy^2 p_x}{2p_s}\Delta B_{y0}(x) \\
& - \Bigg(\frac{qp_x p_y^2 y^2}{2p_s^3}\Delta B_{y0}(x) + \frac{qp_y y^3}{6p_s}\frac{\partial \Delta B_{y0}(x)}{\partial x} \\
& + \frac{qp_x^2 p_y y^3}{3p_s^3}\frac{\partial \Delta B_{y0}(x)}{\partial x} + \frac{qp_x y^4}{12p_s}\frac{\partial^2 \Delta B_{y0}(x)}{\partial x^2} \\
& + \frac{qp_x^3 y^4}{24p_s^3}\frac{\partial^2 \Delta B_{y0}(x)}{\partial x^2} \Bigg) \\
& + \Bigg(\frac{qp_x p_y^4 y^2}{2p_s^5}\Delta B_{y0}(x) + \frac{qp_y^3 y^3}{6p_s^3}\frac{\partial \Delta B_{y0}(x)}{\partial x} \\
& + \frac{2qp_x^2 p_y^3 y^3}{3p_s^5}\frac{\partial \Delta B_{y0}(x)}{\partial x} + \frac{5qp_x p_y^2 y^4}{24p_s^3}\frac{\partial^2 \Delta B_{y0}(x)}{\partial x^2} \\
& + \frac{qp_x^3 p_y^2 y^4}{4p_s^5}\frac{\partial^2 \Delta B_{y0}(x)}{\partial x^2} + \frac{qp_y y^5}{60p_s}\frac{\partial^3 \Delta B_{y0}(x)}{\partial x^3} \\
& + \frac{7qp_x^2 p_y y^5}{120p_s^3}\frac{\partial^3 \Delta B_{y0}(x)}{\partial x^3} + \frac{qp_x^4 p_y y^5}{30p_s^5}\frac{\partial^3 \Delta B_{y0}(x)}{\partial x^3} \\
& + \frac{qp_x y^6}{240p_s}\frac{\partial^4 \Delta B_{y0}(x)}{\partial x^4} + \frac{qp_x^3 y^6}{240p_s^3}\frac{\partial^4 \Delta B_{y0}(x)}{\partial x^4} \\
& + \frac{qp_x^5 y^6}{720p_s^5}\frac{\partial^4 \Delta B_{y0}(x)}{\partial x^4} \Bigg).
\end{aligned}
$$

The lowest order term is just the classical edge focusing in a bending magnet. However, even that contains more information than the classical result: the full field profile should be used, not just the linear part. In fact, if one is only interested in the effect on the tunes, the first term gives the complete effect to lowest order in ΔB_{y0}. One can even get some rudimentary nonlinear effects from that term. One can easily generate higher order terms if desired.

One could perform a similar computation using a multipole representation of the field, or in a curvilinear geometry. It turns out that the first term is correct even for nonzero horizontal curvature. It is not clear whether or not that is true for the higher order terms. The results will be different for a multipole expansion than for the midplane expansion shown above, even though the two representations give the same field in the body of the magnet where there is no longitudinal variation of the field. The choice of expansion must depend on the symmetries which are expected in the magnet construction.

CONCLUSIONS

We have shown how to compute the effect of fringe fields to lowest order in the body field strength and to arbitrary order in the transverse variables. The effect is independent of the longitudinal profile of the end field, but does depend on the manner in which the field expansion is performed (which is related to the construction symmetry of the magnet).

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