LINEAR VLASOV ANALYSIS FOR STABILITY OF A BUNCHED BEAM*

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Abstract

We study the linearized Vlasov equation for a bunched beam subject to an arbitrary wake function. Following Oide and Yokoya, the equation is reduced to an integral equation expressed in angle-action coordinates of the distorted potential well. Numerical solution of the equation as a formal eigenvalue problem leads to difficulties, because of singular eigenmodes from the incoherent spectrum. We rephrase the equation so that it becomes non-singular in the sense of operator theory, and has only regular solutions for coherent modes. We report on a code that finds thresholds of instability by detecting zeros of the determinant of the system as they enter the upper-half frequency plane, upon increase of current. Results are compared with a time-domain integration of the nonlinear Vlasov equation with a realistic wake function for the SLC damping rings. There is close agreement between the two calculations.

INTRODUCTION

We consider coherent instabilities in longitudinal motion of a bunched beam in a storage ring. This problem is usually treated by the linearized Vlasov equation, although recently a time-domain integration of the nonlinear Vlasov equation has also been employed [1]. The linearized equation has been cast in various ways as an integral equation expressing mode coupling. Oide and Yokoya [2] made an important advance when they linearized the Vlasov equation about the proper equilibrium distribution determined by Haïssinski's equation. They then transformed to action–angle coordinates of the corresponding distorted potential well, and took Fourier transforms in time and angle to derive an integral equation. A discretization of their equation has been applied by several authors to find the current threshold of a microwave instability [3, 4, 5, 6]. Some success in agreement of thresholds with tracking studies has been reported, but some difficulties have been noticed as well, by the present authors and other investigators [3].

The Oide-Yokoya equation has the formal appearance of an ordinary linear eigenvalue problem for the frequency ω, and the procedure followed in [2] was to discretize the equation in a straightforward way, then find the eigenvalues and eigenvectors numerically. This leads to difficulties arising from the presence of the "incoherent spectrum" in the spectrum of eigenvalues, that is, the continuously distributed, real, amplitude-dependent frequencies of the underlying single particle motion. In the discretization, the number of eigenvalues is the dimension of the matrix, and almost all of the eigenvalues are merely trying to imitate the continuous incoherent spectrum, becoming more densely distributed as the dimension of the matrix is increased. The discrete eigenvalue of a coherent mode at the threshold of instability approaches degeneracy with a value in the imitated continuous spectrum, making it difficult to separate the desired discrete mode. Moreover, the procedure does not have proper convergence as the dimension of the matrix increases, since the eigenvectors of the exact problem are generalized functions (of the form delta function plus principal value), analogous to Case - van Kampen modes. In principle, these cannot be represented numerically, although in the numerical solution the eigenvectors indeed look like vague imitations of the expected generalized function.

A way to avoid these difficulties was proposed in Ref. [7], where it was shown that a simple change in the choice of the unknown function eliminates consideration of the continuous spectrum. Instead of a linear eigenvalue problem one has a nonlinear function of the frequency, the determinant of the new integral equation, and zeros of this function in the upper half plane correspond to unstable coherent modes. This formulation was derived through considerations of functional analysis, by looking for an equation with compact operator. It could have been derived as well by steps in analogy to coasting beam theory. The coasting beam theory can also be formulated as a linear eigenvalue problem, but the conventional and more suitable formulation is in terms of a nonlinear function of frequency (the dispersion function), quite analogous to our determinant.

REGULARIZED INTEGRAL EQUATION

We follow the notation and formulation of Ref.[7]. The angle-action coordinates for motion in the distorted potential well are (ϕ, J), and the frequency of that motion is $\Omega(J)$. The Haïssinski equilibrium distribution is $f_0(J)$. The perturbation to the equilibrium is $f_1(\phi, J, \tau)$, with time coordinate $\tau = \omega_s t$. A Fourier transform in ϕ and Laplace transform in τ gives $\hat{f}_1(m, J, \omega)$, the variable conjugate to τ being $-i\omega$ with $\mathrm{Im}\,(\omega) > 0$. The unknown function to be determined by our integral equation is

$$g(m, J, \omega) = e^{J/2}(\omega - m\Omega(J))\hat{f}_1(m, J, \omega)\ , \qquad (1)$$

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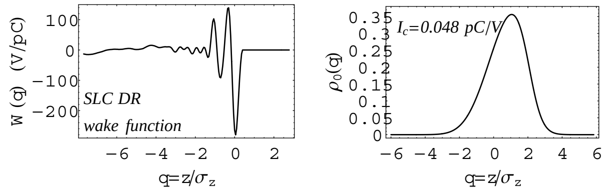

Figure 1: Wake function for the SLC Damping Ring and Haïssinski distribution for $I_c = 0.048$ pC/V.

whereas the unknown for Oide-Yokoya is $\hat{f}_1$. Our equation has the following form derived in [7]:

$$g(m, J, \omega) - ie^{J/2}\check{f}_1(m, J, 0) + \sum_{m'=-\infty}^{\infty} \int_0^\infty dJ' \frac{H(m, J, m', J')g(m', J', \omega)}{\omega - m'\Omega(J')} = 0\,, \tag{2}$$

where $\check{f}_1$ is the initial value of $f_1(m, J, \tau)$. The kernel is given in terms of the wake function $W(q)$ and the canonical transform $q \to Q(\phi, J)$, where the normalized coordinate $q = z/\sigma_z$ is the distance to the synchronous particle over the nominal bunch length, positive at the front of the bunch. We have

$$H(m, J, m', J') = -\frac{I_c f_0'(J)e^{(J-J')/2}}{2\pi} \int d\phi \sin m\phi \int d\phi' \cos m'\phi' \cdot Q_1(\phi, J) W(Q(\phi, J) - Q(\phi', J'))\,, \tag{3}$$

where $Q_1 = \partial Q/\partial\phi$ and $I_c = e^2 N/(2\pi\nu_s\sigma_E)$ is a normalized current parameter. Here N is the bunch population, ν_s the synchrotron tune, and σ_E the nominal energy spread.

APPLICATION OF THE METHOD

To discretize Eq.(2) we first change the integration variable from J' to $y = (J')^{1/2}$, then replace the integration by a numerical quadrature rule on a uniform mesh $\{y_i\}$. By appropriate pole subtractions we take care to make the quadrature rule valid for $\mathrm{Im}\ \omega = v$ positive but arbitrarily small. With N_J mesh points and a truncation of the mode sum at $m' = N_m$, the equation takes the form

$$[\mathbf{1} + \mathbf{A}(\omega)]g = h\,, \tag{4}$$

where **1+A** is a square matrix of dimension $N_J N_m$, and g is a vector with components $g(m, J(y_i), \omega)$. We report calculations with $N_J = 40,\ N_m = 12$.

The matrix $\mathbf{A}(\omega)$ is analytic for $\mathrm{Im}\ \omega > 0$, since it consists of sums of products of discretized integrals that have that property. If the determinant $D(\omega) = \det[\mathbf{1} + \mathbf{A}(\omega)]$ has a zero in the upper half plane at $\omega = \hat{\omega}$, then the solution g of (4) will have a pole at $\hat{\omega}$ for any initial value term h. By the inverse Laplace transform the distribution function $f_1(m, J, \tau)$ then grows exponentially in time as

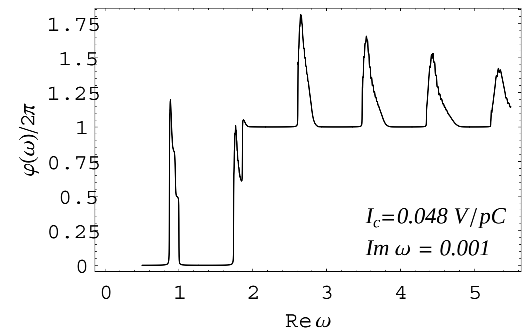

Figure 2: Phase of determinant $D(\omega)$ *vs.* Re ω for fixed Im ω.

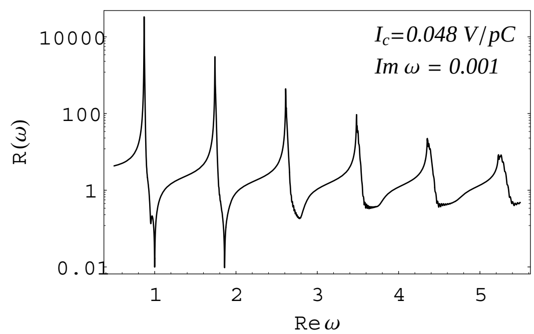

Figure 3: Absolute value of determinant $D(\omega)$ *vs.* Re ω for fixed Im ω.

$e^{\mathrm{Im}\ \hat{\omega}\tau}$. Since no instability is expected at low current, there must be a critical current, or threshold, at which a pole first appears in the upper half plane when the current is increased from a small initial value.

A convenient way to detect zeros entering the upper plane is based on the fact that the number of zeros in the half plane $\mathrm{Im}\ \omega > v$ is $n = (1/2\pi i)\int_\Gamma d\omega D'(\omega)/D(\omega)$, where the closed contour Γ consists of the line $\mathrm{Im}\ \omega = v$ plus the semi-circle at infinity. (We assume that no zero is on the line $\mathrm{Im}\ \omega = v$). Now $D'(\omega)/D(\omega) = d\log D(\omega)/d\omega$, and $D(\omega) = 1 + \mathcal{O}(1/\omega)$, $D'(\omega) = \mathcal{O}(1/\omega^2)$ at infinity. Consequently, $2\pi n$ is the total change in the phase of D along the line $\mathrm{Im}\ \omega = v$.

We have applied this analysis to longitudinal motion in the SLC Damping Ring, a case in which the wake function is fairly well known, thanks to calculations of K. Bane. His calculated wake is shown in the left picture of Fig. 1. The nominal bunch length is $\sigma_z = 5.58$ cm. That figure also reports the profile of the Haïssinski distribution for current parameter $I_c = 0.048$ corresponding to 1.71×10^{10} particles/bunch. The same value of current was used in the calculations yielding the results shown in Fig 2 through 5. The plot of the phase $\varphi(\mathrm{Re}\ \omega + iv)/2\pi$ in Fig 2 suggests that $I_c = 0.048$ is above threshold since the phase increases by 2π around $\mathrm{Re}\ \omega$=2, indicating the presence of a pole with $\mathrm{Im}\ \hat{\omega} > 0.001$. Consistently, in correspondence to this value the plot of $R(\omega) = |D(\omega)|$ shows a dip. While other local minima are present in the plot for $R(\omega)$ only the one close to Re $\omega \simeq 1$ suggests the possible existence of an additional pole. We used the location of these two local minima to initialize Newton searches for the roots of $D(\omega)$. The search starting from Re $\omega \simeq 1$ failed to converge whereas the other yielded the root $\hat{\omega} = 1.860 + .00231i$ - apparently the only root for the current under consideration.

Next, we determined the mode associated to the pole at $\hat{\omega}$, identified as the eigenvector of the matrix $\mathbf{A}(\hat{\omega})$ corresponding to eigenvalue -1. We evaluated the corresponding perturbation to the bunch distribution $f_1(\phi, J, \hat{\omega}) = \sum_m \hat{f}_1(m, J, \hat{\omega}) \exp(im\phi)$ with the help of Eq. (1). The resulting density plot is given in Fig. 4. The mode is largely dominated by the $m = 2$ term but small contributions from the $m = 1$ and $m = 3$ harmonics are also present. Not surprisingly, the peak value of the distribution is located at $J = J_* = 3.03$ with Re $\hat{\omega} - 2\Omega(J_*) = 0$, where the denominator $|\hat{\omega} - 2\Omega(J)|$ is smallest.

We checked the results of this analysis against numerical solutions of the full Vlasov equation in the time domain using the method of Ref.[1]. We tracked the relative r.m.s energy spread σ_p starting from equilibrium. Since the numerical equilibrium and Vlasov integration are not exact, there is a growing envelope of σ_p indicating that the equilibrium is unstable (thick band in Fig. 5). We fitted σ_p to the form $\kappa(\tau) = A + B \exp(\tau \text{Im } \hat{\omega}) \cos(\tau \text{Re } \hat{\omega})$ to estimate oscillation frequency and growth rate. We found $\hat{\omega} = 1.862 + .00229i$, remarkably close to the result of the linearized Vlasov analysis. Because beyond 100 synchrotron periods some saturation of the instability appears to take place we limited the fit to the numerical data from the first 50 periods. A plot of the fitting function against the Vlasov solver over a selected time interval is shown in the inserted picture in Fig. 5.

Finally, we repeated the calculation by changing the current. The values of current yielding an unstable frequency $\hat{\omega}$ together with the resulting growth rates are reported in Fig. 6. A linear extrapolation from the two values with smallest rates gives $I_c^{th} = 0.0432$ as an estimate for the current threshold.

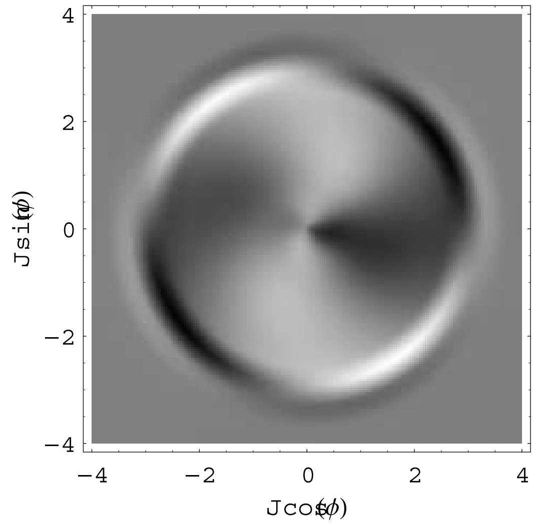

Figure 4: Density plot of $f_1(m, J, \hat{\omega})$, the unstable mode for $I_c = 0.048$ with frequency $\hat{\omega} = 1.860 + i2.311 \times 10^{-3}$.

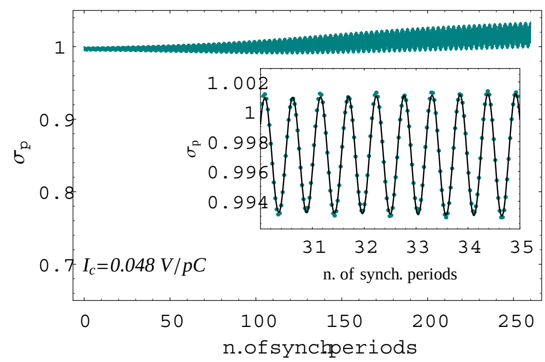

Figure 5: Evolution of relative energy spread above current threshold from Vlasov solver in the time domain. Inserted picture: best fit to determine growth rate. One synchrotron period corresponds to $\Delta\tau = 2\pi$.

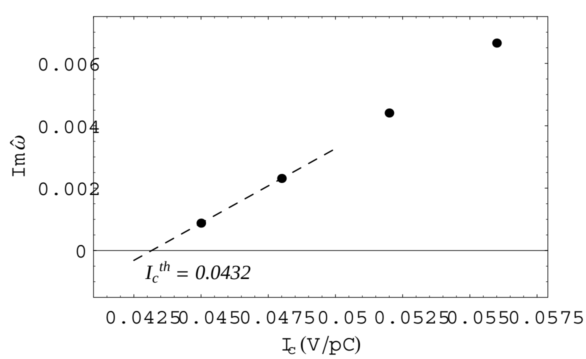

Figure 6: Growth rate Im $\hat{\omega}$ of unstable mode as function of current parameter I_c.

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