MISMATCH OSCILLATIONS IN HIGH-CURRENT ACCELERATORS

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INTRODUCTION

In an important paper, Struckmeier and Reiser [1] studied the oscillation frequencies of mismatched beams assuming precise knowledge of the phase advances σ_0 and σ. In the present paper we start instead with the quadrupole focusing strength, beam line charge and emittance. With these input quantities, the smooth approximation gives substantial errors. Our simple modification improves the accuracy by a factor of five at $\sigma_0 = 83°$.

REVIEW: MATCHED-BEAM CASE

Previously, we analyzed the envelope equations for a quadrupole-focused K-V beam by expanding the ripple in a small parameter ε (of the order of the focusing strength) and using integrations [2]. Given the beam current, emittance, and field strength, we found the envelopes a(z) and b(z) and obtained $\langle a_0 \rangle$, a_{max}, and the phase advances σ and σ_0 with explicit results for various truncations of the expansion. The zeroth-order results corresponded to those from the smooth approximation [3]. The first-order results, involving the simplest correction terms, gave 3 to 10 times improvement in accuracy, e.g., to ~1% at $\sigma_0 = 70°$.

We treated a non-relativistic beam, easily generalized to β of order unity. The beam was assumed uniform and transported by linear quadrupoles having symmetric but otherwise arbitrary profiles. The paraxial equations for the envelopes a and b are:

$$a'' = -K(z)a + \frac{\in^2}{a^3} + \frac{2Q}{a+b},$$

$$b'' = +K(z)b + \frac{\in^2}{b^3} + \frac{2Q}{a+b}.$$

As in [2], K(z) represents the alternating gradient and $\in$ the emittance (we assume $\in_x = \in_y$). Q is the normalized perveance, defined non-relativistically by

$$Q = (4\pi\varepsilon_0)^{-1}(m/2q)^{1/2} I V^{-3/2},$$

with m the ion mass, I the beam current, qV the ion energy, and ε_0 the vacuum dielectric constant.

These envelope equations have matched solutions $a_0(z)$ and $b_0(z)$ with period 2L, the full length of a quadrupole cell. Denoting averages over 2L by angle brackets, we have $\langle b_0 \rangle = \langle a_0 \rangle = A$, with A the mean matched radius.

The matched solutions a_0 and b_0 can be expressed in terms of A and small ripples $\rho_a(z)$ and $\rho_b(z)$:

$$a_0(z) \approx A[1 + \rho_a(z)],$$

$$b_0(z) \approx A[1 + \rho_b(z)],$$

with $\rho_b(z)$ identical to $\rho_a(z)$ except for displacement by length L. The leading-order ripple $\rho_0(z)$ is

$$\rho_0(z) = -\iint K; \qquad (1)$$

further terms are discussed in Ref. [2]. As before, $\int$ and $\iint$ refer to indefinite integrals of periodic functions with lower limits chosen to make average values over 2L vanish. For symmetric cases, the inner integral starts at the midpoint of a quadrupole and the outer integral at $L/2$.

Ref. [5] shows that, with only 0.04% error at $\sigma_0 = 120°$,

$$a_0 + b_0 \to 2A.$$

A is obtained from the matching equation [5]:

$$K_{eff\dagger} = \frac{Q}{A^2} + \in_\dagger^2/A^4 \qquad \text{with} \qquad (2a)$$

$$K_{eff\dagger} \equiv K_{eff}(1 + c\Phi), \qquad (2b)$$

$$K_{eff} \equiv \left\langle [\textstyle\int K]^2 \right\rangle, \qquad (2c)$$

$$\in_\dagger^2 \equiv \in^2\left(1 + \Phi + \tfrac{5}{2}\Phi^2\right), \qquad (2d)$$

$$\Phi \equiv 3\left\langle [\textstyle\iint K]^2 \right\rangle. \qquad (2e)$$

The confining force $K_{eff\dagger}$ is mean square of the *integral* of the focusing force. The small factor c is typically 0.08 [2].

PERTURBED-BEAM CASE

Consider small perturbations of the above matched solutions a_0 and b_0:

$$a(z) = a_0(z) + x(z), \qquad x \ll \langle a_0 \rangle \qquad (3)$$

$$b(z) = b_0(z) + y(z), \qquad y \ll \langle b_0 \rangle. \qquad (4)$$

Putting Eqs. (3) and (4) into the envelope equations and linearizing,

$$x'' = -K(z)x - \frac{3\in^2}{a_0^4}x - \frac{2Q}{(a_0 + b_0)^2}(x+y), \qquad (5)$$

$$y'' = +K(z)y - \frac{3\in^2}{b_0^4}y - \frac{2Q}{(a_0 + b_0)^2}(x+y). \qquad (6)$$

Expanding and retaining terms to order ε^2, we get

$$s'' = -\left(K(z) - \frac{12\in^2}{A^4}\rho_0(z)\right)d - \frac{3\in^2}{A^4}\left(1 + 10\rho_0^2\right)s - \frac{Q}{A^2}s, \qquad (7)$$

$$d'' = -\left(K(z) - \frac{12\in^2}{A^4}\rho_0(z)\right)s - \frac{3\in^2}{A^4}\left(1 + 10\rho_0^2\right)d, \qquad (8)$$

where

$$s = x + y, \qquad (9)$$

$$d = x - y \qquad (10)$$

are sum and difference functions corresponding roughly to breathing and quadrupole excitations.

Since $\rho_0 = -\iint K$ and since A can be eliminated by solving the matching equation (2a), Eqs. (7) and (8) depend only on our input quantities $K(z)$, $\in$, and Q.

SOLUTION OF PERTURBED EQUATIONS

Computer studies show that for σ_0 and σ less than about 80°, and for σ_0 as large as 180° with σ's appropriately chosen, the s or d modes can be separately excited, with relatively small amplitude in the other mode. We use the term "mode" in these nearly-decoupled, well-behaved, stable cases. As seen in Fig. 1a, the excited mode can be fairly sinusoidal. Our goal here is to analyze the frequencies of these quasi-sinusoidal oscillations.

In Eqs. (7) and (8) the coefficients of the first terms on the right generally have strong oscillations, while the coefficients of the other terms are constant or have no fundamental Fourier component. In order to solve Eqs. (7) and (8) we make the *ansatz* that the well-behaved modes are driven mainly by the oscillations at the fundamental period so that all harmonics may be neglected. For example, we keep only the fundamental component of the focusing term: $K(z) \to K_1 \cos(\pi z/L)$.

We look for the frequency of the s mode by putting

$$s = \cos(u\pi z/L) \tag{11}$$

with u to be determined. We insert the above into Eq. (8), integrate twice, then iterate. The final result is

$$\sigma_s^2 = (2L)^2\left[2K_{eff} + 2\left(\in^2/A^4\right)(1+\Phi)\right](1-\Phi)^{-1}. \tag{12}$$

with $\sigma_s \equiv 2\pi u$ the sum mode phase advance. Similarly,

$$\sigma_d^2 = (2L)^2\left[K_{eff}(1+\tfrac{1}{3}\Phi) + 3\left(\in^2/A^4\right)(1+\tfrac{8}{9}\Phi)\right](1-\Phi)^{-1}. \tag{13}$$

Some results from Eqs. (12) and (13)—with A from Eq. (2)—are plotted in Fig. 2. It shows that Eqs. (12) and (13) closely match the results obtained by numerical integration. For the last points (at σ_0 = 179°, σ = 89°), we obtain σ_s = 225° and σ_d = 199° with accuracy of 1.5% and –1.6%, respectively.

COMPARISON OF RESULTS

In Refs. [2] and [5] we showed higher-order expressions for σ_0 and σ. By neglecting the Φ corrections, those equations become the smooth approximations

$$\sigma_{0_{smooth}}^2 = (2L)^2 K_{eff}, \tag{14}$$

$$\sigma_{smooth}^2 = (2L)^2 \in^2/A_{smooth}^4, \tag{15}$$

with A_{smooth} obtained from the standard matching equation, which is our Eq. (2) with $\Phi \to 0$:

$$K_{eff} = Q/A_{smooth}^2 + \in^2/A_{smooth}^4. \tag{16}$$

Equation (2c) defines K_{eff}. Equation (14) appears in Ref. [4] as (10.92) while Eq. (15) is Eq. (6) in Ref. [3] (with different notations).

Dropping Φ corrections in Eqs. (12) and (13) and using (14) and (15), we have

$$\sigma_{s_{smooth}}^2 = 2\sigma_{0_{smooth}}^2 + 2\sigma_{smooth}^2, \tag{17}$$

$$\sigma_{d_{smooth}}^2 = \sigma_{0_{smooth}}^2 + 3\sigma_{smooth}^2, \tag{18}$$

in agreement with Ref. [1].

However, these results are in error for large σ_0—e.g., by 18 and 24%, respectively, at σ_0 = 199°. Our correction terms improve the accuracy there by over a factor of ten.

Figure 4 in Ref. [1] shows curves stated to "represent the smooth approximation results" with high accuracy for $\sigma_0 \le 90°$. Actually, those curves were obtained from exact numerical values of σ_0 and σ, not from smooth approximations – which would be (14) and (15) in our notation. Ref. [1] plots what we will call hybrid values:

$$\sigma_{s_{hybrid}}^2 = 2\sigma_{0_{exact}}^2 + 2\sigma_{exact}^2, \tag{19}$$

$$\sigma_{d_{hybrid}}^2 = \sigma_{0_{exact}}^2 + 3\sigma_{exact}^2. \tag{20}$$

We use the term "hybrid" because these are not exact formulas even though they employ exact numerical values for σ_0 and σ. Eqs. (19, 20) are useful if one already knows those exact values, but when the input data is in the form of quadrupole voltage, emittance, and current, the smooth approximation would invoke the zero-order Eqs. (14–18). Much better accuracy is obtained from our Eqs. (12, 13).

The three alternatives are compared in Fig. 3 for quadrupole strength giving an exact σ_0 of 83.4°. Results from the smooth-approximation formulas do not come close to the exact results anywhere. Our formulas improve the accuracy by a factor of five for the full range of σ obtained by adjusting current and emittance.

For σ_0 > 90° there is an unstable region, such as that shown in Fig. 4. For an exact σ_0 of 112.2°, eigenvalue analysis [1] predicts instability for 44.5° < σ < 86.3°. Beyond this unstable zone there are regions where the mismatch oscillations are well behaved (Fig. 1a), satisfying our model, and regions where the opposite is the case (Fig. 1b). Fig. 4a shows that for σ around 89°, where the regular decoupled oscillations satisfy our ansatz, our formulas give very good accuracy for σ_s and σ_d. In the case of σ = 109°, with poorly behaved oscillations, the results are less accurate but still a major improvement over the smooth approximation (Fig. 4b). The hybrid formula, of course, approaches perfection at the far right of Fig. 4c, where $\sigma \to \sigma_0$, since the exact σ_0 is given in advance.

For extreme quadrupole strengths producing large σ_0, such as the 179° case mentioned above, the narrow regions where oscillations are decoupled and regular may be of limited practical interest, but the accuracy of our formulas there seems to vindicate our ansatz. For $\sigma_0 \le 80°$ or so, our model is good for any combination of current and emittance so that equations (12) and (13) could be useful in practical applications.

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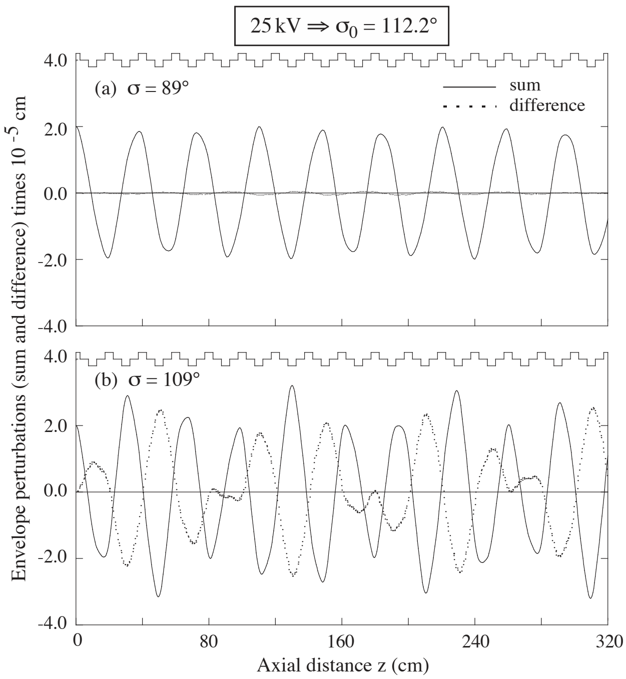

Figure 1: Envelope perturbations: (x+y) and (x–y) vs. z for initial perturbations $x(0)=y(0)=10^{-5}$ cm, $x'(0)=y'(0)=0$. Voltage of 25 kV gives $\sigma_0 = 112.2°$. (a) Current and emittance are adjusted to give $\sigma = 89°$; sum and difference modes are well decoupled (dotted line has tiny amplitude); therefore Eqs. (12) and (13) are accurate, as confirmed in Fig. 3. (b) Current is decreased to give $\sigma = 109°$; strong coupling between modes violates our model. Fig. 4 shows decreased accuracy.

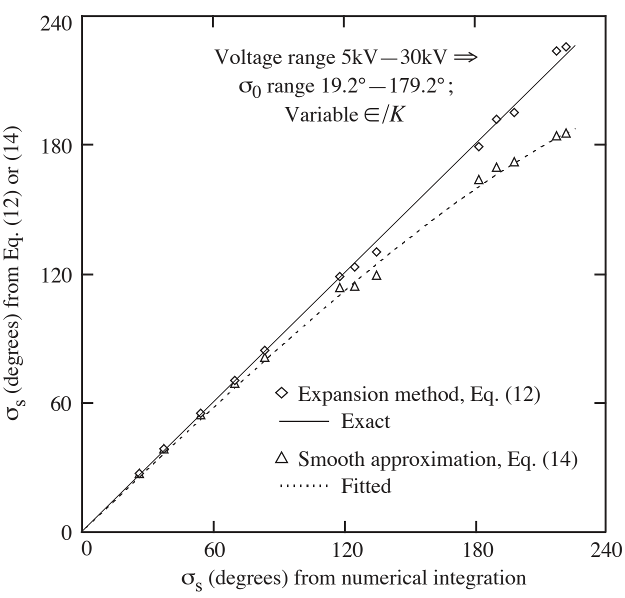

Figure 2: Even-mode phase advances σ_s for various σ_0 and σ. Voltage range 5–30 kV. (Odd-mode results—not shown here—are similar.) Abscissa values obtained by counting mismatch oscillations for well-behaved cases (cf. Fig. 1a). Smooth-approximation (17) errors approach 18%, but values from Eq. (12) lie close to the straight line representing zero error. For each σ_0 up to about 80°, Eq. (12) was applied over a wide range of current and emittance and therefore of σ. However, for larger σ_0, some σ's gave ill-behaved fluctuations (cf. Fig. 1b), preventing measurement of σ_s as well as invalidating our model. Only measurable cases are shown.

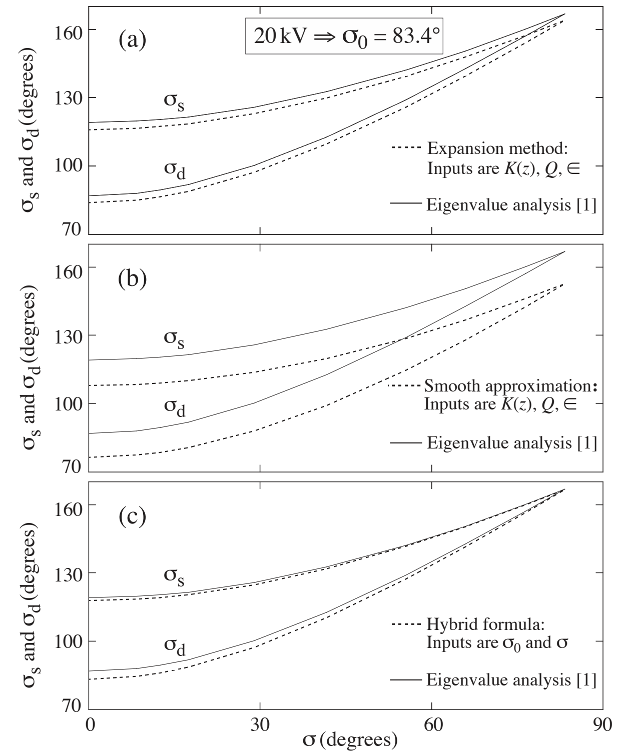

Figure 3: σ_s and σ_d vs. σ for $\sigma_0 = 83.4°$. Solid curves are from numerical integrations and eigenvalue analyis, as in Ref. [1]. (a) Eqs. (12) and (13) are good to 2% for all σ. (b) Smooth approximation Eqs. (17), (18) are off by about 10%. (c) Hybrid formulae (19, 20) require input of exact values of σ and σ_0 [1].

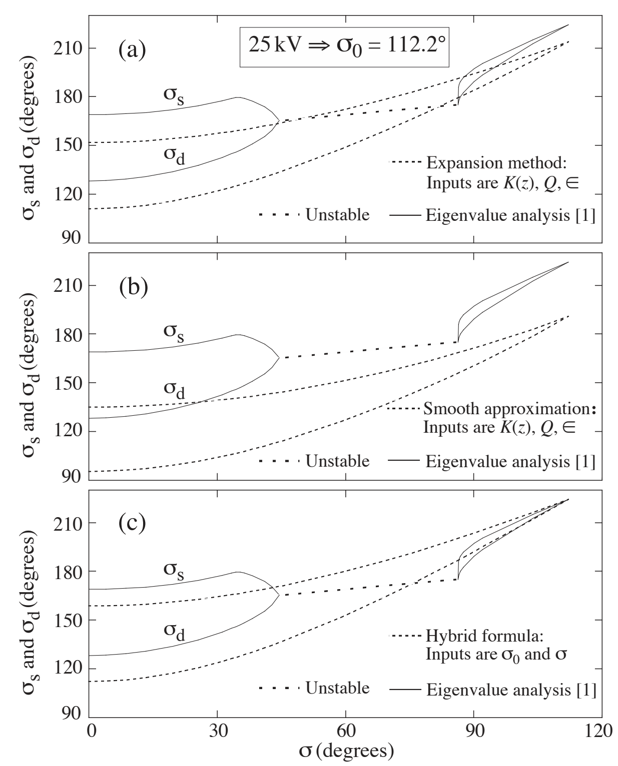

Figure 4: Same as Fig. 3 except $\sigma_0 = 112.2°$. Eigenvalue analysis (Ref. [1]) and Eqs. (12), (13) are inaccurate for $0 \le \sigma < \sim 86.3°$, where s and d oscillations are strongly coupled. (a) Eqs. (12, 13) are good near 89°, with oscillations nearly sinusoidal. (b) Smooth approximation shows poor accuracy everywhere. (c) Hybrid formula, using exact σ and σ_0, is automatically exact for $\sigma = \sigma_0$.