ELECTROMAGNETIC FIELDS OF AN OFF-AXIS BUNCHED BEAM IN A CIRCULAR PIPE WITH FINITE CONDUCTIVITY AND THICKNESS - II

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Abstract

The general exact solution exploited in [1] is applied, introducing suitable dimensionless parameters, and using appropriate asymptotic limiting forms, to compute the wake field multipoles for the different paradigm cases of LHC and DAFNE.

INTRODUCTION

In [1] we computed the fields of a (bunched) beam in a pipe with walls of finite conductivity and thickness, for the simplest pipe-geometry (circular). We solved the problem by computing the Fourier transform of the wake potential Green's function produced by a point particle running at constant velocity $\beta c\hat{u}_z$, at a distance r_o off axis of a circular cylindrical pipe with radius b, wall conductivity σ and thickness Δ.

The solution found is exact but complicated, so that in most cases of practical interest one has to resort to suitable limiting forms. In this paper we introduce a number of asymptotic approximations appropriate, in particular, to LHC (Large Hadron Collider) and DAFNE, whose relavant figures are collected in Tables I and II.

THE GREEN'S FUNCTION

In [1] we obtained the Green's function for an off-axis point particle running parallel to the axis of a circular pipe of radius b with finite conductivity σ and thickness Δ, viz.:

$$\tilde{G}_m(k,r,r_0) = \tilde{G}_m^{\infty}(k,r,r_0) + \frac{q_o}{2\pi\epsilon_o}\frac{I_m(k'r_0)I_m(k'r)}{bk'I_m(k'b)}\frac{N(k)}{D(k)}, \quad (1)$$

where

$$\tilde{G}_m^{\infty}(k,r,r_0) = \frac{q_o}{2\pi\epsilon_o}\left\{A(k,r,r_0) - \frac{I_m(k'r_0)}{I_m(k'b)}K_m(k'b)I_m(k'r)\right\}. \quad (2)$$

In Eq. (2) $k' = k/\gamma$, $\tilde{G}_m^{\infty}$ is the solution of the wave equation corresponding to the perfectly conducting pipe, $A(\cdot)$,$N(k)$ and $D(k)$ are defined in [1] and

$$\eta = \frac{Z_o\sigma - ik\beta}{ik\beta} \quad (3)$$

where $Z_0 = (\mu_0/\epsilon_0)^{1/2}$ is the free space wave impedance. Eq. (1) reduces to the solution obtained in [2] in the limit $\Delta \to \infty$ of an infinitely thick wall.

ASYMPTOTIC APPROXIMATIONS

In most cases of practical interest, one may resort to suitable (asymptotic) limiting forms, since many problem-specific (dimensionless) parameters are either very large or very small.

Large parameters

The following inequality always holds in view of the assumed beam spectral features:

$$|\bar{k}b| = \left|\sqrt{k'^2 - i\sigma\beta kZ_o}\right| b \sim \left|\sqrt{-i\sigma\beta kZ_o}\, b\right| \equiv \left|\frac{b}{\delta_{wall}}\right| \gg 1, \quad (4)$$

where

$$\delta_{wall} = (-i\sigma\beta kZ_o)^{-1/2} \quad (5)$$

is the electromagnetic skin depth. One has also $|\bar{k}d| \gg 1$, since $d \gtrsim b$. Note also that, within the useful spectral ranges discussed above, one has from Eq. (3):

$$\eta \simeq -i\frac{Z_o\sigma}{k\beta}. \quad (6)$$

Accordingly, using the well known large-argument forms of the (modified) Bessel functions:

$$I_m(z) \sim \frac{e^z}{\sqrt{2\pi\, z}}, \quad K_m(z) \sim \sqrt{\frac{\pi}{2z}}e^{-z}, \quad (7)$$

for $I_m(\cdot)$ and $K_m(\cdot)$ with arguments $\bar{k}b$ and $\bar{k}d$ in Eq. (1), one gets a simpler form for both $N(k)$ and $D(k)$, viz.:

$$N(k) = -\bar{k}^2K_m'(k'd)\sinh\bar{k}\Delta + \eta k'\bar{k}K_m(k'd)\cosh\bar{k}\Delta, \quad (8)$$

$$D(k) = \sinh\bar{k}\Delta\left[k'^2\eta^2 I_m(k'b)K_m(k'd) - \bar{k}^2 I_m'(k'b)K_m'(k'd)\right] + \eta k'\bar{k}\cosh\bar{k}\Delta\left[I_m'(k'b)K_m(k'd) - I_m(k'b)K_m'(k'd)\right]. \quad (9)$$

The relative error stemming from use of Eq.s (8), (9) in Eq. (1) is shown in Fig.1 as a function of kb, for the lowest order multipoles, in the special case (worst admissible one for LHC Table I) $\sigma = 5.7\cdot 10^7\Omega^{-1}m^{-1}$, $\gamma = 5\cdot 10^2$ and $b = 1.5cm$. The absolute error within the spectral range of interest is $\sim\ 10^{-6} \div 10^{-7}$.

Small parameters

Let us next discuss the asymptotic limit:

$$|k|b/\gamma \sim |k|d/\gamma \ll 1. \quad (10)$$

For reasons which will be clarified soon, it is convenient to discuss separately the monopole $(m = 0)$ and multipole $(m \geq 1)$ terms.

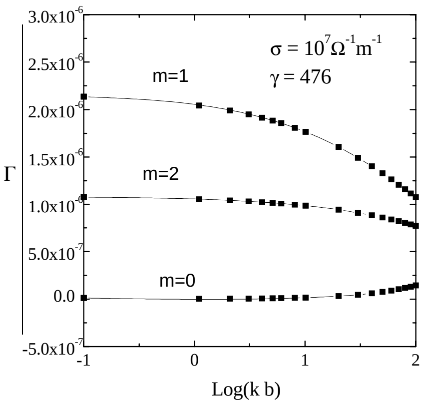

Figure 1: Relative error Γ on $(2\pi\epsilon_0/q)[\tilde{G}_m - \tilde{G}_m^{\infty}]$ versus kb after assuming $\bar{k}b \gg 1$ and using Eq.s (8), (9) in place of Eq. (1); monopole, dipole and quadrupole terms (m=0,1,2).

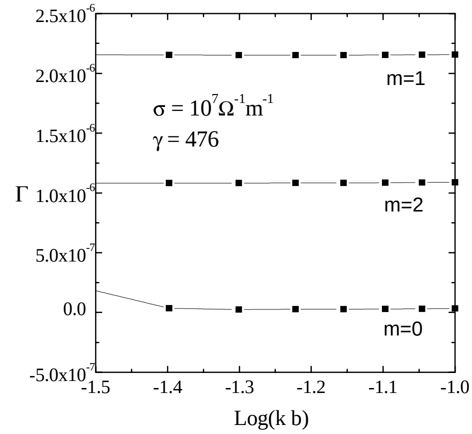

Figure 2: Relative error Γ on $(2\pi\epsilon_0/q)[\tilde{G}_m - \tilde{G}_m^{\infty}]$ versus kb after assuming $kb \ll 1$ and using Eq.s (16), (21) in place of Eq. (1); monopole, dipole and quadrupole terms (m=0,1,2).

The Monopole Term $(m = 0)$ In the limit Eq. (10), one uses the zero-th order modified Bessel functions approximation valid for small arguments [3]:

$$I_0(\zeta) \sim 1, \quad K_0(\zeta) = -log(\zeta), \tag{11}$$

and hence the monopole term in Eq. (1) using Eq.s (8),(9) can be written

$$\tilde{G}_0(k,r,r_0) = \tilde{G}_0^{\infty}(k,r,r_0) + \frac{q_o}{2\pi\epsilon_o}\frac{\gamma^2}{bk^2}\left[\frac{b}{2} + \eta\,\delta_{wall}\,\coth(\Delta/\delta_{wall})\right]^{-1}. \tag{12}$$

For a *very thick pipe wall*, $|\bar{k}\Delta| \sim |\Delta/\delta_{wall}| \gg 1$, whence $|\coth(\Delta/\delta_{wall})| \sim 1$, and Eq. (12) becomes:

$$\tilde{G}_0(k,r,r_0) = \tilde{G}_0^{\infty}(k,r,r_0) + \frac{q_o}{2\pi\epsilon_o b}\frac{\gamma^2}{k^2}\left(\frac{b}{2} + \eta\delta_{wall}\right)^{-1} \tag{13}$$

which, in the further limit (appropriate, e.g., both for LHC and DAFNE):

$$\left|2\eta\frac{\delta_{wall}}{b}\right| \gg 1, \tag{14}$$

yields the known result [2]:

$$\tilde{G}_0(k,r,r_0) = \tilde{G}_0^{\infty}(k,r,r_0) + \frac{q_o\beta\gamma^2}{2\pi\epsilon_o b}(1+i)\sqrt{\frac{\beta}{2\sigma Z_o k}}. \tag{15}$$

For a *finite-thickness pipe wall*, $|\,\Delta/\delta_{wall}\,| \geq 1$, in the same limit Eq. (14), Eq. (12) yields:

$$\tilde{G}_0(k,r,r_0) = \tilde{G}_0^{\infty}(k,r,r_0) + \frac{q_o\beta\gamma^2}{2\pi\epsilon_o b}(1+i)\sqrt{\frac{\beta}{2\sigma Z_o k}}\tanh(\Delta/\delta_{wall}). \tag{16}$$

This latter, in the limit of infinite wall thickness, $|\Delta/\delta_{wall}| \to \infty$, gives back Eq. (15). The relative error produced by using Eq. (16) in place of Eq. (1) is shown in Fig.2 as a function of kb.

Multipole Terms ($m \geq 1$) In the asymptotic limit $|k|b/\gamma \ll 1$, $|k|d/\gamma \ll 1$ one uses in Eq.s (2), (1), (8) and (9) the small-argument asymptotic form of the modified Bessel functions of m-order [3]:

$$I_m(\zeta) \sim \left(\frac{\zeta}{2}\right)^m \frac{1}{m!},$$

$$K_m(\zeta) \sim \frac{(m-1)!}{2}\left(\frac{\zeta}{2}\right)^{-m}, \quad (m > 0). \tag{17}$$

Hence, from (2):

$$\tilde{G}_m^{\infty}(r,r_0) \approx \tilde{G}_m^{free\ space}(r,r_0) - \frac{q_o}{2\pi\epsilon_o}\frac{1}{2m}\left(\frac{rr_o}{b^2}\right)^m, \tag{18}$$

where

$$\tilde{G}_m^{free\ space}(r,r_0) \approx \frac{q_o}{2\pi\epsilon_o}\frac{1}{2m}\left(\frac{rr_o}{b^2}\right)^m R(r,r_o), \tag{19}$$

$$R(r,r_o) = \begin{cases} (r_0/r)^m & r_0 \leq r \leq b, \\ (r/r_0)^m\ , & r \leq r_o, \end{cases} \tag{20}$$

and, from Eq.s (1),(8) and (9):

$$\tilde{G}_m(k,r,r_0) = \tilde{G}_m^{\infty}(k,r,r_0) + \frac{q_o}{2\pi\epsilon_o}\frac{1}{m}\left(\frac{rr_o}{b^2}\right)^m \cdot$$

$$\cdot\left[1 + \frac{k^2\eta b}{m\bar{k}\gamma^2}\tanh(\bar{k}\Delta)\frac{\dfrac{k^2\eta d}{m\bar{k}\gamma^2} + \coth(\bar{k}\Delta)}{\dfrac{k^2\eta d}{m\bar{k}\gamma^2} + \tanh(\bar{k}\Delta)}\right]^{-1}. \tag{21}$$

which, using Eq.s (5), (6) can be equally written:

$$\tilde{G}_m(k,r,r_0) = \tilde{G}_m^{\infty}(k,r,r_0) + \frac{q_o}{2\pi\epsilon_o}\frac{1}{m}\left(\frac{rr_o}{b^2}\right)^m \cdot \tag{22}$$

$$\cdot\left[1 + \frac{b/\delta_{wall}}{m\beta^2\gamma^2}\tanh\left(\frac{\Delta}{\delta_{wall}}\right)\frac{\dfrac{d/\delta_{wall}}{m\beta^2\gamma^2} + \coth\left(\dfrac{\Delta}{\delta_{wall}}\right)}{\dfrac{d/\delta_{wall}}{m\beta^2\gamma^2} + \tanh\left(\dfrac{\Delta}{\delta_{wall}}\right)}\right]^{-1}.$$

The relative error produced by using Eq. (21) in place of Eq. (1) for m=1,2 is shown in Fig.2.

As expected, the error increases with kb, but remains very small throughout the meaningful spectral range. Similar to the monopole term case, for a *very thick pipe wall* , one has $|\bar{k}\Delta| \sim |\Delta/\delta_{wall}| \gg 1$, and hence $\sinh \bar{k}\Delta \sim \cosh \bar{k}\Delta$. Thus Eq. (21) becomes:

$$\tilde{G}_0(k,r,r_0) = \tilde{G}_0^{\infty}(k,r,r_0) + \frac{q_o}{2\pi\epsilon_o m}\left(\frac{rr_o}{b^2}\right)^m \left(1 + \frac{b/\delta_{wall}}{m\beta^2\gamma^2}\right)^{-1}. \tag{23}$$

The *finite-thickness pipe wall* , $|\Delta/\delta_{wall}| \geq 1$ case, will be now discussed with reference to a number of limiting cases relevant to our applications.

LHC In the Large Hadron Collider one has:

$$\left|\frac{b/\delta_{wall}}{\beta^2\gamma^2}\right| \ll 1, \qquad \left|\frac{d/\delta_{wall}}{\beta^2\gamma^2}\right| \ll 1. \tag{24}$$

Accordingly, for not-too-small values of $|\Delta/\delta_{wall}|$,

$$\tilde{G}_m(k,r,r_o) = \tilde{G}_m^{\infty}(k,r,r_o) + \frac{q_o}{2\pi\epsilon_o}\frac{1}{m}\left(\frac{rr_o}{b^2}\right)^m \left[1 - \frac{b/\delta_{wall}}{m\beta^2\gamma^2}\coth\left(\frac{\Delta}{\delta_{wall}}\right)\right]. \tag{25}$$

Equation (25) reproduces the limit form of Eq. (23) under Eq. (24) provided $\Delta \gg |\delta_{wall}|$. In the extreme limiting case $|\Delta/\delta_{wall}| \ll 1$ the expression in square brackets in Eq. (21) becomes simply $(1 + b/d)^{-1}$, so that using (18), one has:

$$\tilde{G}_m(r,r_0) \approx \tilde{G}_m^{free\ space}(r,r_0) - \frac{q_o}{2\pi\epsilon_o}\frac{1}{2m}\left(\frac{rr_o}{b^2}\right)^m \left[1 - 2\left(1 + \frac{b}{d}\right)^{-1}\right] \tag{26}$$

which reduces to the free-space term, if $\Delta \to 0$, i.e. $d \to b$, as expected.

Ultrashort Bunch Machines In ultrashort bunch machines, including, e.g., DAFNE, one has (Table II):

$$\left|\frac{b/\delta_{wall}}{m\beta^2\gamma^2}\right| \gg 1. \tag{27}$$

Accordingly, for not-too-small values of $|\Delta/\delta_{wall}|$,

$$\tilde{G}_m(k,r,r_0) = \tilde{G}_m^{\infty}(k,r,r_0) + \frac{q_o}{2\pi\epsilon_o}\left(\frac{rr_o}{b^2}\right)^m \beta^2\gamma^2\frac{\delta_{wall}}{b}\coth\left(\Delta/\delta_{wall}\right). \tag{28}$$

In the extreme limiting case $|\Delta/\delta_{wall}| \ll 1$ the expression in square brackets in Eq. (22) becomes simply $(1+b/d)^{-1}$, so that using (18), one has:

$$\tilde{G}_m(r,r_0) \approx \tilde{G}_m^{free\ space}(r,r_0) - \frac{q_o}{2\pi\epsilon_o}\frac{1}{2m}\left(\frac{rr_o}{b^2}\right)^m \left[1 - 2\left(1 + \frac{b}{d}\right)^{-1}\right] \tag{29}$$

which reduces to the free-space term, if $\Delta \to 0$, i.e. $d \to b$, as expected.

LHC Design parameters

Nominal Circumference L_c	26658 m
Number of bunches N_b	2835
Bunch length σ_s	$(7 \div 13)cm$
Lorentz factor γ	500÷7000
Pipe diameter	3 cm
Wall thickness	50 μm (Cu) + 1 mm (SS)
Wall conductivity	$(5.7 \cdot 10^7 \div 10^{10})\Omega^{-1}m^{-1}$
Circulation frequency	11.2455 kHz

Table I

DAFNE Design parameters

Nominal Circumference L_c	97.69 m
Number of bunches N_b	120
Bunch length σ_s	$2cm$
Lorentz factor γ	1000
Pipe diameter	10 cm
Wall thickness	2 mm (Al)
Wall conductivity	$3.4 \cdot 10^7\Omega^{-1}m^{-1}$
Circulation frequency	368.26 MHz

Table II

REFERENCES

[1] R. P. Croce, Th. Demma, S. Petracca, ”Electromagnetic Fields of an Off-Axis Bunch in a Circular Pipe with Finite Conductivity and Thickness”, these Proceedings.

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[3] M. Abramowitz, A. Stegun *Handbook of Mathematical Functions*, Dover, 1965.