NON GAUSSIAN TRANSVERSE DISTRIBUTIONS IN A STOCHASTIC MODEL FOR BEAM HALOS

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Abstract

We face, in a stochastic framework, the problem of the halo formation in high intensity beams of charged particles. We sum up our recent results on this argument, and in the final part of the paper we introduce non Gaussian distributions as good candidates to account for sensible escape of particles from the density core.

INTRODUCTION

In high intensity beams of charged particles, proposed in recent years for a wide variety of accelerator–related applications, it is very important to keep at low level the beam loss to the wall of the beam pipe, since even small fractional losses in a high–current machine can cause exceedingly high levels of radioactivation. However, numerical simulations, and a few experiments at high current proton linacs, suggest that an extended, low–density halo is formed around the beam: it is then necessary to obtain a more quantitative understanding of the physics of the halo, in order to control and reduce its undesirable effects. In recent papers [1, 2] we faced this problem in the framework of a stochastic approach to the collective beam dynamics. Charged particle beams are usually described either in terms of classical dynamical systems, i. e. by a collisionless plasma with the dynamics of a suitable phase space, or by a Fokker–Planck equation in the phase–space variables. In our alternative approach [3] we propose instead to describe the collective dynamics of the particle beams by a stochastic *and* time–reversal invariant description in configuration space. This picture is effective in the regime of stability, where a balance is realized between the energy dissipation and the external energy pumping. The sample paths of the representative particle are described by a diffusive stochastic differential equation ruled by the beam emittance, and the two coupled stochastic–hydrodynamic equation describing the dynamics of the beam density profile and the motion of its center are derived by a stochastic variational principle [4]. The two coupled equations are also equivalent to a unique linear equation of the form of a (mesoscopic) Schrödinger equation, with the universal Planck constant replaced by the emittance of the beam. Our approach, starting by an apparently different point of view, formally converges then to the so–called quantum–like approach to beam dynamics [5]. In the framework of our scheme is also possible to implement an active control for the beam dynamics [6], an aspect which could be useful in order to eliminate halos. In references [1, 2] we have adapted our scheme to a preliminary study of the problem of halo formation and control, and we have described the collective, not deterministically controllable effects by diffusive (Gaussian) fluctuations. In the present paper we make the hypothesis that non–Gaussian processes, for example the family of the Student distributions, with longer tails and non-zero jump probability could provide a better understanding of the long–distance particle behavior.

STOCHASTIC MODEL FOR HALOS

The coupled stochastic–hydrodynamic equations derived by the stochastic variational principle [4] and describing the particle beam dynamics in the stability regime [3] take the form , respectively, of a continuity equation:

$$\partial_t \rho = -\nabla \cdot (\rho \mathbf{v}) \,, \tag{1}$$

and of an evolution (Madelung) equation for the conservative dynamics

$$\partial_t S + \frac{m}{2}\mathbf{v}^2 - 2mD^2 \frac{\nabla^2 \sqrt{\rho}}{\sqrt{\rho}} + V(\mathbf{r}, t) = 0 \,, \tag{2}$$

where , if m is the mass of a particle, the drift velocity and the (Jacobi-Madelung) function S are connected by the gradient form $m\mathbf{v} = \nabla S$, and the diffusion coefficient is proportional to the beam emittance α: $\alpha = 2mD$. Eqns. (1) and (2) describe the collective behavior of the beam at each instant of time through the evolution of both its particle density and its velocity field. Introducing the representation $\psi = \sqrt{\rho}\, \mathrm{e}^{iS/\alpha}$, the two coupled real equations (1) and (2) are equivalent to a single complex Schrödinger equation for the function ψ, with the Planck action constant $\hbar$ replaced by the unit of action α:

$$i\alpha\partial_t \psi = -\frac{\alpha^2}{2m}\nabla^2\psi + V\psi \,. \tag{3}$$

In this "Schrödinger-like" (Sl) formulation the phenomenological "wave function" ψ carries the information on the dynamics of both the beam density and the beam velocity field, since the velocity field is determined via the gradient relation by the phase function S.

Space–charge effects

Among the various possible sources of halos, in [1] we have taken into account the spreading effects due to the presence of a space–charge distribution in the framework of our approach. We have implemented the Sl-equation (3)

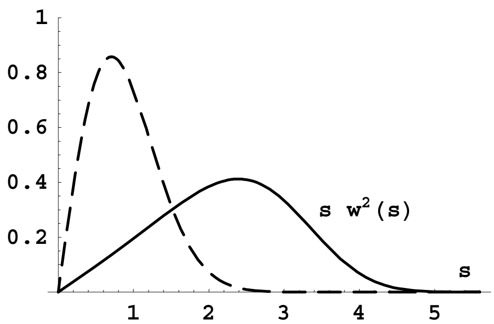

Figure 1: The radial distribution $s\,w^2(s)$ compared with the distribution in absence of space charge (dashed line) for a strong space-charge strength . All quantities are dimensionless.

in a cylindrically symmetric configuration and coupled it with the Maxwell equations for the electromagnetic field generated by the charge distribution. We have thus obtained two nonlinear coupled equations. Considering the stationary distribution of the particle density, we have obtained that the density profile associated to the presence of space–charge is sensibly spread with respect to the Gaussian profile obtained if space–charge is absent (see Fig. 1). The distribution exhibits a larger dispersion, and we have shown that, for sufficiently intense charge density, we can find up to 10^5 particles per meter beyond a distance of 10σ (with σ denoting the width of the Gaussian core), at variance with the zero particles in absence of charge effects.

Stationary halo distributions

The solutions discussed in the previous subsection have been obtained through a numerical approach, because the nonlinearly coupled equations do not allow for exact analytical solutions. In the same reference [1] we have then also employed a variational technique to obtain an approximate analytical version of the numerical solution for the density profile. In fact, an analytic expression for the density profile leads to analytical forms for the drift velocity and the generating potential, which in turn are needed if one wish to engineer a controlling mechanism to eliminate the halo. We have then optimized a reasonable test form for the density profile, and we have found that the variational solutions show the presence of a node in the density profile, and a corresponding singularity in the potential. A node can be the signature for the presence of a ring–shaped halo, well separated by the core; in fact, it is completely open the question if halo is due to long tails in the density distribution, or if a true halo must be separated from the core. On the other hand, we have obtained a tail by space–charge effects, which however are not the only possible source of halo. Then, in the spirit of a preliminary inquiry about both the interpretations, we have reversed our point of view; we have in fact explicitly introduced a realistic ring distribution, and we have used the dynamical equations to obtain information on the characteristics of the associated potential and of the associated velocity fields. The potential exhibits an extra relative maximum if compared with the harmonic one, while in correspondence the velocity field exhibits a "bump" with respect to the straight (linear) behavior.

Dynamical control of the halo

In reference [2] we have extended our analysis to the problem of the halo dynamics, considering both the the halo elimination and its reformation. This last problem cannot be accounted for in a pure Quantum–like (Ql) description since, because of the stochastic extremal principle, a change in the distribution produces a change in the dynamics. On the other hand the Ql dynamics leads to distributions which are stable attractors; thus, the halo, if scraped away, will be restored in characteristic times, but through a not extremal Fokker–Planck evolution. We have then accordingly modified our scheme: the velocity field changes slowly on the relaxation time scales, while the evolutions are ruled only by a Fokker–Planck equation until the balance has been restored. We got an estimate of the time τ needed by a non stationary, halo–free distribution to relax toward the stationary distribution with a halo when the dynamics is supposed frozen in the configuration that produces this halo: we have obtained $\tau \approx 10^{-8} \div 10^{-7}\mathrm{sec}$ [2]. Since this relaxation time depends on the parameters of our beam, a comparison with possibly measured phenomenological times could constitute a good check on the soundness of the model. We have also begun to analyze possible transitions from a beam with halo toward a halo–free one, putting an emphasis on the possible dynamics which allows this halo elimination.

NON GAUSSIAN DISTRIBUTIONS

Non gaussian stationary states

Values of the *maximum-to-RMS* ratio as large as 10 or 12 have been observed in simulations and it is generally agreed that this is called halo (see [8] p. 286). Hence there are good reasons to try non gaussian transverse distributions with longer tails so that the probability of finding particles far away from the beam axis becomes larger. Generalized Student distributions $\Sigma(\nu; a^2)$

$$f(x) = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{\nu}{2}\right)}\,\frac{a^\nu}{(a^2+x^2)^{\frac{\nu+1}{2}}}, \quad \nu > 0. \tag{4}$$

are good candidates. They have finite variance σ^2 only for $\nu > 2$ and $a^2 = (\nu-2)\sigma^2$. Moreover $\Sigma(\nu, (\nu-2)\sigma^2)$ converges in distribution to $\mathcal{N}(0, \sigma^2)$ for $\nu \to +\infty$. The radial control potential for a stationary cylindrical beam:

$$V_\nu(r) = \frac{\alpha^2}{2m\sigma^2}\,r^2\,(\nu+2)\,\frac{\frac{r^2}{\nu-2} + (\nu+10)\frac{\sigma^2}{4}}{\left[r^2+(\nu-2)\sigma^2\right]^2}$$

tends to be harmonic for $\nu \to +\infty$, and is constant beyond a certain radius. This possibly points to some effect due to the presence of the conducting walls.

Lévy driven processes

The trajectories of a process with a stationary Student $\Sigma(\nu, (\nu-2)\sigma^2)$ ground state can be simulated from the Gaussian driven SDE

$$dX(t) = v_\nu(X(t))dt + \sqrt{2\mathcal{D}}\, dW(t)$$

with suitable velocity field. On the other hand a different kind of process can be generated by independent increments with Student law (4) through a SDE

$$dX(t) = \beta(X(t))dt + \gamma\, dS(t)$$

where $dS(t)$ is distributed according to $\Sigma(\nu; a^2)$, and $\beta(x)$ is a suitable velocity field. No complete theory is available for a Lévy driven stochastic mechanics.

The Student distributions $\Sigma(\nu; a^2)$ are infinitely divisible (Lévy distributions) [9]. Hence they are possible limit distributions of row-sums of r.v.'s, but different from the gaussian. As every infinitely divisible law, the $\Sigma(\nu; a^2)$ are possible distributions of the increments of a Markov process with independent increments (Lévy process) [10]. Non-gaussian Lévy processes have moving discontinuities: only gaussian Lévy processes have almost every trajectory everywhere continuous [11]. The standard example of a non-gaussian Lévy process is the compound Poisson process. The rate of occurrence, and the height of the moving discontinuities are regulated by the Lévy function $L_t(x)$ characteristic of a given process with independent increments [11]. However the Lévy function of a Student $\Sigma(\nu; a^2)$ process is not explicitly known.

Discontinuities in these processes can be used to describe the occasional escape of particles out from an otherwise well collimated beam in an accelerator. Moreover, by suitably tuning the parameters (a and ν) we can also keep the transverse distribution of the beam reasonably close to a gaussian. We can contrast the trajectories for both a Gauss and a Student process with comparable dispersions. In the Figures 2, 3 a few examples of the Student trajectories are produced. Since for fixed a the Student variance (which diverges for $0 < \nu \leq 2$) is a decreasing function of ν, it is apparent that also the jump probability and their length will generally decrease with ν. The Figures for $\nu > 2$ (finite, decreasing variance) are selected among the possible cases by excluding all the trajectories showing an almost gaussian behavior. For $\nu > 2$ several trials are needed before getting a trajectory definitely drifting away from the beam core.

In this model we suppose that a Student-Lévy noise will be active only inside the beam, while beyond a given distance from the core the process will be driven by a Gauss noise; moreover the velocity field attracts the trajectory back toward the core only in its neighborhood, while beyond a given distance the process is just a free diffusion. Remark that in this example the Gaussian trajectories would be essentially that of an Ornstein–Uhlenbeck process.

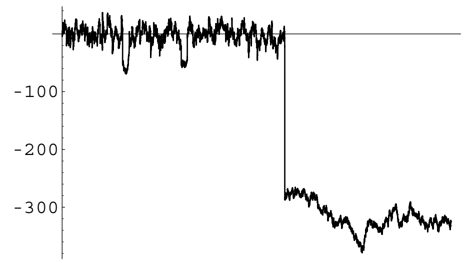

Figure 2: Student process with $\nu = 2.0$

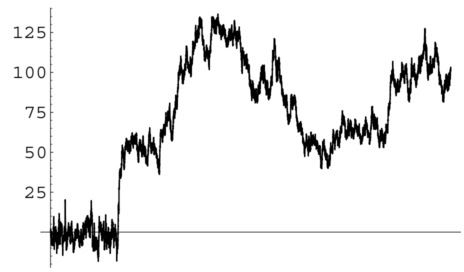

Figure 3: Student process with $\nu = 3.0$

We conclude by remarking that in our opinion for a halo model the possibility of rare, long jumps in the trajectories is more relevant than just the modest increase in the variance produced by the Student noise with respect to the Gauss (almost everywhere continuous) noise.

REFERENCES

[1] N. Cufaro Petroni, S. De Martino, S. De Siena, and F. Illuminati, Phys. Rev. ST Accel. Beams **6**, 034206 (2003).

[2] N. Cufaro Petroni, S. De Martino, S. De Siena, and F. Illuminati, Int. Journal of Mod. Phys. B **18**, 607 (2004).

[3] N. Cufaro Petroni, S. De Martino, S. De Siena, and F. Illuminati, Phys. Rev. E 63, 016501 (2001)

[4] F. Guerra and L. M. Morato, Phys. Rev. D **27**, 1774 (1983); E. Nelson, *Quantum Fluctuations* (Princeton University Press, Princeton N. J., 1985).

[5] R. Fedele, G. Miele and L. Palumbo, Phys. Lett. A **194**, 113 (1994), and references therein; S. I. Tzenov, Phys. Lett. A **232**, 260 (1997).

[6] N. Cufaro Petroni, S. De Martino, S. De Siena, and F. Illuminati, J. Phys. A **32**, 7489 (1999).

[7] N. Cufaro Petroni, S. De Martino, and S. De Siena, Phys. Lett. A **245**, 1 (1998).

[8] T. Wangler, *RF Linear accelerators* (Wiley, New York 1998).

[9] L. Bondesson, *Generalized gamma distributions and related classes of distributions and densities* (Springer, New York 1992).

[10] W. Feller, *An introduction to probability theory and its applications - vol. II* 2nd edition (Wiley, New York 1971).

[11] M. Loève, *Probability theory - vol, II* 4th edition (Springer, New York 1978).