

LINEAR COUPLING THEORY OF HIGH INTENSITY BEAMS

G. Franchetti, * I. Hofmann, GSI, Darmstadt, Germany
 M. Aslaninejad, IPM, Tehran, Iran

Abstract

Linear coupling with space charge effects in coasting beams is studied by using the second order moment equations. A coherent shift of the resonance condition and nonlinear saturation effects in the emittance transfer result from space charge. Crossing through the linear coupling resonance from below allows full emittance exchange, whereas the exchange is found to be largely suppressed by space charge effects for crossing from above.

INTRODUCTION

Linear coupling for high-intensity beams is of practical importance in synchrotrons like the SIS18 at GSI, where it may be desirable to eliminate the strong imbalance of transverse emittances before extraction. In the absence of space charge, linear coupling by skew quadrupoles is well-understood [1, 2, 3]. Space charge effects, on the other hand, have recently been shown in a linearized analytical study to be significant [4, 5] as they lead to a modification of the single particle resonance condition $Q_x - Q_y = N$ (N harmonic of skew).

MODEL

A consistent study must go beyond a linearized theory, therefore we present here a fully self-consistent modelling based on the complete second order moment equations derived by Chernin [6], which couple all second order moments of a beam, including the “even” ($\langle xx \rangle$, $\langle xx' \rangle$, etc.) as well as the “odd” ($\langle xy \rangle$, $\langle xy' \rangle$, etc.) coupling moments, with $' \equiv d/ds$. We use these equations and follow the notation by Chernin assuming v is the four-component vector (x, x', y, y') . A 4×4 matrix of second order moments Σ is defined by $\Sigma_{i,j} \equiv \langle v_i v_j \rangle - \langle v_i \rangle \langle v_j \rangle$, also

$$M \equiv \begin{pmatrix} 0 & 1 & 0 & 0 \\ -\tilde{k}_x & 0 & -\tilde{j} & 0 \\ 0 & 0 & 0 & 1 \\ -\tilde{j} & 0 & -\tilde{k}_y & 0 \end{pmatrix}. \quad (1)$$

The matrix elements contain external and space charge force terms: $\tilde{k}_x = k_x - q_{xx}$, $\tilde{k}_y = k_y - q_{yy}$, and $\tilde{j} = j - q_{xy}$, with k_x, k_y the horizontal and vertical focusing, j the external linear coupling, and the (nonlinear) space charge defocusing q_{xx} , q_{yy} and q_{xy} . The time-dependence of Σ is then given by

$$\Sigma' = M\Sigma + (M\Sigma)^T, \quad (2)$$

* g.franchetti@gsi.de

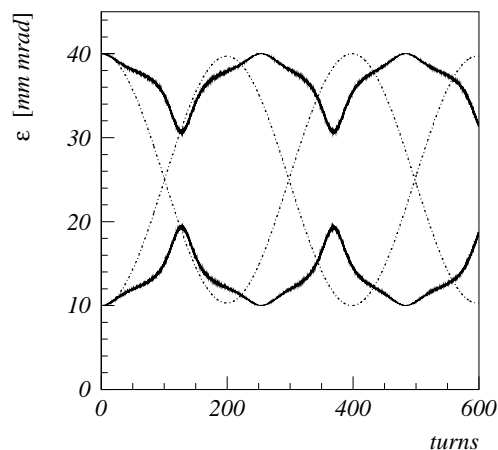


Figure 1: Rms emittance evolution for $Q_{x0} = 4.16$ compared with zero-space-charge case for $Q_{x0} = 4.2$ (dotted).

where T denotes the transposed matrix. These equations become the well-known envelope equations, if the coupling terms are set to zero.

We approximate the real lattice by a constant focusing lattice and the linear coupling by a single thin lens kick generating all harmonics. For initializing the numerical calculation we use a solution that has been matched in the absence of skew with initial $\epsilon_x = 40\pi$ mm m-rad and $\epsilon_y = 10\pi$ mm m-rad. We use a fixed vertical machine tune of $Q_{y0} = 3.2$, and initial vertical space charge tune shift of $\Delta Q_y = -0.2$ – typical values for the SIS18 (216 m circumference). The coupling is given in terms of N_s as the number of turns required for a full emittance exchange exactly on the resonance, ignoring space charge.

APPLICATIONS

We study the case of split tunes ($N = 1$) related to the SIS18, and assuming $N_s = 200$. The maximum possible emittance transfer – far from a complete exchange – occurs for $Q_{x0} = 4.16$. The corresponding time dependence is shown in Fig. 1, along with the familiar case of zero space charge with full exchange exactly on the resonance at $Q_{x0} = 4.2$. In both cases the exchange process is a periodic one, but space charge limits the exchange of emittances to only 10π mm m-rad for this relatively weak

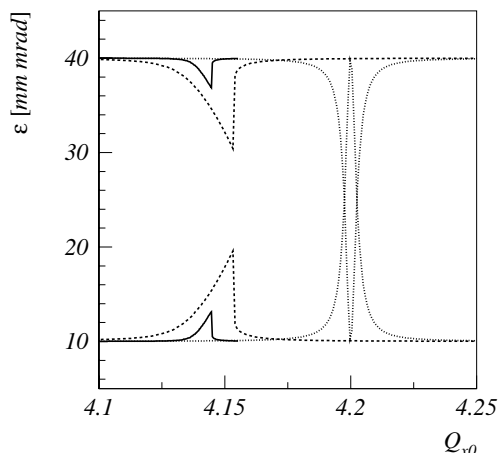


Figure 2: Maximum achievable rms emittance exchange for $N = 1$ as function of Q_{x0} , with $N_s = 200$ (dashed) and $N_s = 1000$ (continuous) compared with zero-space-charge case (dotted, $N_s = 200$).

coupling.

We show in Fig. 2 the maximum emittance exchange for two different values of N_s as function of Q_{x0} (kept constant during each simulation run). As expected, the zero-space-charge case shows full emittance exchange exactly on the resonance at $Q_{x0} = 4.2$. The effect of space charge is: (1) a downwards shift of the resonant tune as predicted in Ref. [5]; (2) a strong asymmetry of the response curve, and (3) a strong reduction of the maximum emittance transfer, depending on the strength of skew. (2,3) are typical nonlinear phenomena, which largely vanish if the space-charge-dominated stop-band in Fig. 2 merges with the pure linear coupling stop-band by either strongly enhanced skew or negligible space charge. In Ref. [5] an analytical calculation of the self-consistent resonance location was derived, with an approximate expression valid in the limit $|\Delta Q_{x,y}| \ll Q_{x,y}$ and $N \neq 0$:

$$Q_x - Q_y = N + |\Delta Q_x| \frac{\epsilon_r - 1}{2(1 + \sqrt{\epsilon_r} Q_{y0}/Q_{x0})}. \quad (3)$$

Here, the second term on the r.h.s. is the coherent resonance shift. Eq. 3 determines the Q_x (from which Q_{x0} follows by matching), where the resonance appears for given initial ϵ_r and in the linearized theory limit $N_s \rightarrow \infty$. Note that we have assumed $\epsilon_r \equiv \epsilon_x/\epsilon_y \geq 1$, otherwise x and y should be inverted. For our example we find $Q_x = 4.05$, or $Q_{x0} = 4.14$, which is confirmed by the simulation results of Fig. 2. Using Eq. 3, together with the matching, leads to a de-tuning. The dynamical reduction of ϵ_r shifts the resonance to larger Q_{x0} and further exchange is stopped. The sharp r.h.s. edge of the stop-band with space charge is typical for nonlinear oscillations – see a similar behavior for the envelope oscillation [7, 8].

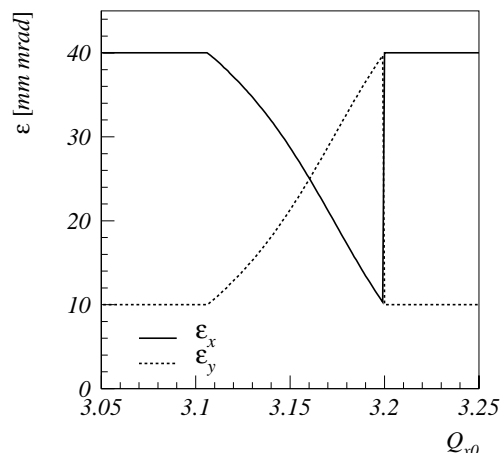


Figure 3: Maximum rms emittance exchange for un-split tunes as function of Q_{x0} without external skew.

The resonance with un-split tunes ($N = 0$) is different. The effect of the spontaneous “self-skewing” instability of the second order “odd” eigen-mode dominates the picture as shown in Fig. 3, where an external skew is absent. As shown in Refs. [4, 9, 10, 5] the exponential instability of this mode is an intrinsic second order feature of a space-charge-dominated beam. The left edge of its stop-band is defined by the condition $Q_x = Q_y$, and the right edge by $Q_{x0} = Q_{y0}$. Comparison with a case where a skew with $N_s = 200$ was applied has shown only a small additional effect. In a full particle-in-cell simulation additional higher order effects (like the fourth-order “Montague resonance”) were found to dominate the picture, but the coupling effect on emittances is similar [4, 9].

For practical applications in the SIS18 we need to study the effects dynamically by slowly shifting the horizontal tune from lower to higher values across the resonance region, which is performed in a linear ramp extending over 10.000 turns. Fig. 4 shows the result for split tunes. The emittance is found to be practically independent of the speed of crossing – as long as very fast crossing is avoided – and of the actual skew strength. It is mainly controlled by the strength of space charge due to the fact that for slow crossing the beam follows the matched solution of the coupled lattice. To show this we have searched for each tune in Fig. 4 a matched solution belonging to identical values of the two relevant kinematic invariants of the coupled lattice [11, 12], and found a nearly perfect match between the dynamical emittance curve and the matched solutions. In Fig. 4 the emittances are found to be equal for $Q_{x0} = 4.186$, where the code gives $Q_x = 4.082$, $Q_y = 3.083$, hence $Q_x - Q_y = 1$ is satisfied to high accuracy; beyond this point emittances switch symmetrically. This is fully consistent with Eq. 3 predicting the

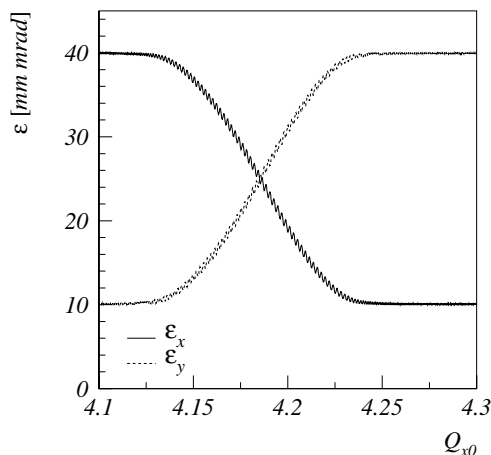


Figure 4: Rms emittances for dynamical stop-band crossing from below with linear tune ramp in time and $N_s = 200$.

absence of the coherent resonance shift for $\epsilon_r = 1$. The self-consistent change of emittances is such that the linear theory resonance stop-band is effectively never crossed, but it moves along with the tune – a kind of “snowplow” effect: for each ϵ_r from Fig. 4 we use Eq. 3 to determine the tune, where the resonance occurs, and find that it agrees with the corresponding tune of Fig. 4 within a relative error $< 10^{-3}$. Hence, Eq. 3 models to good accuracy the tune - emittance relationship for slow crossing, which agrees with the matched solution of the coupled system. The exchange in the absence of space charge, for comparison, is modelled differently [2].

The nonlinear nature of the space charge effect leads to a quite different result, if the resonance is crossed from above (split tunes, $\epsilon_r > 1$), see Fig. 5. Exchange of emittances is not found, since the system “jumps” across the resonance starting at a tune consistent with the sharp edge found in Fig. 2 for stationary tunes. A slightly decreased Q_{x0} results in a reduction of ϵ_r , which causes a shift of the stop-band to the right according to Eq. 3 up to the point, where the tune has left the resonance region. This counter-motion therefore suppresses emittance exchange, which is most effective for narrow stop-bands as in Fig. 2, hence significantly stronger skew is needed to bring the final emittances closer together or even get exchange. We find that a strong skew with $N_s \approx 10$ enforces almost full exchange, but this results in pronounced mismatch oscillations.

CONCLUSION

We have found that the combined effect of space charge and linear coupling gives rise to new phenomena. Earlier derived analytical conditions for resonance and instability are confirmed in the linear small-signal regime. But they

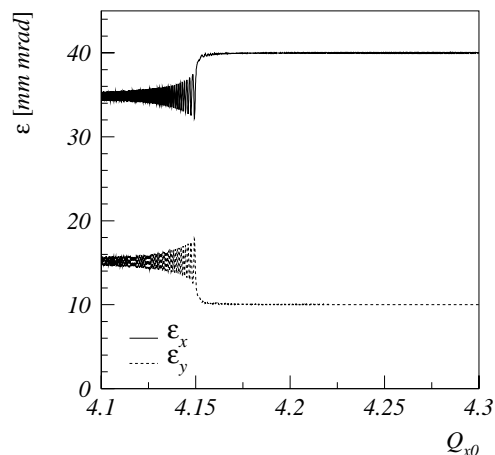


Figure 5: Rms emittances for dynamical stop-band crossing from above ($N_s = 200$).

are significantly modified by space charge in the nonlinear regime, which causes a strong asymmetry between crossing from below and above. Synchrotron motion in bunched beams may wash out some of the coherence effects and requires a separate study using fully 3D particle-in-cell simulation.

REFERENCES

- [1] K. Schindl and P. van der Stock, Technical Report No. CERN/PS/BR 76-19, CERN (unpublished).
- [2] E. Metral, Technical Report No. CERN-PS-2001-066-AE, CERN (unpublished).
- [3] D. A. Edwards and L. C. Teng, IEEE Trans. Nucl. Sci. **NS-20**, Vol. 3, 885 (1973)
- [4] I. Hofmann, Phys. Rev. E **57**, 4713 (1998).
- [5] M. Aslaninejad and I. Hofmann, Phys. Rev. ST Accel. Beams **6**, 124202 (2003).
- [6] D. Chernin, Part. Accel. **24**, 29 (1988).
- [7] F. Sacherer, Ph.D. thesis, University of California, Lawrence Radiation Lab Report UCRL-18454, 1968 (unpublished).
- [8] A. V. Fedotov and I. Hofmann, Phys. Rev. ST Accel. Beams **5**, 024202 (2002).
- [9] I. Hofmann and O. Boine-Frankenheim, Phys. Rev. Lett. **87**, 034802 (2001).
- [10] I. Hofmann, G. Franchetti, O. Boine-Frankenheim, J. Qiang and R. D. Ryne, Phys. Rev. ST Accel. Beams **6**, 024202 (2003).
- [11] A. J. Dragt, F. Neri, and G. Rangarajan, Phys. Rev. A **45**, 2572 (1992).
- [12] R. A. Kishek, J. J. Barnard, and D. P. Grote, IEEE Particle Accelerator Conference (PAC 99), New York, March 29 - April 2, 1999, p. 1761.