STABILITY DIAGRAMS FOR LANDAU DAMPING WITH TWO-DIMENSIONAL BETATRON TUNE SPREAD FROM BOTH OCTUPOLES AND NONLINEAR SPACE CHARGE APPLIED TO THE LHC AT INJECTION

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Abstract

The joint effect of space-charge nonlinearities and octupole lenses is discussed for the case of a quasi-parabolic distribution function in the two transverse planes, considering a monochromatic beam and neglecting the longitudinal variation of the transverse space-charge forces. The self-consistent nonlinear space-charge tune shift corresponding to the above distribution function is first derived analytically. Noting that a reasonable approximation of the space-charge tune shift is given considering only linear terms in the betatron action variables, the dispersion relation is solved analytically in this approximate case. As expected, in the absence of external (octupolar) nonlinearities, the result of Möhl and Schönauer is recovered: there is no stability region. In the absence of space charge, the stability diagrams of Berg and Ruggiero are recovered. The new result is applied to the LHC at injection.

INTRODUCTION

The influence of space-charge nonlinearities on the Landau damping mechanism of transverse coherent instabilities has been studied thirty years ago by Möhl and Schönauer for coasting and rigid bunched beams [1]. Later Möhl extended these results to head-tail modes in bunched beams [2]. The basic results of these studies are that in the absence of external (octupolar) nonlinearities, the space-charge nonlinearities have no effect on bean stability, as the incoherent space-charge tune spread moves with the beam. When octupoles are added, the incoherent space-charge tune spread is "mixed-in", and in this case the octupole strength required for stabilization can depend strongly on the sign of the excitation current of the nonlinear lenses.

In a workshop in 1999 [3], the community was not entirely comfortable with the interpretation by Möhl and Schönauer and recommended further theoretical studies and controlled measurements.

The dispersion relation found and solved in this paper is discussed in the first section. It is the same as the one of Refs. [1,2], solved here for the quasi-parabolic distribution $f\,(J_x, J_y) \propto [1-(J_x+J_y)\,/\,(5\sigma^2)\,]^2$. This distribution function leads to a physical beam profile, which is a bell-shaped curve smoothly going to zero, together with its first and second derivative, at $\sim\pm 3.2\,\sigma$. The self-consistent space-charge tune shift is given in the second section. The exact dispersion relation has been expressed in Ref. [4], and remains to be solved. In the third section, the dispersion relation is solved analytically for the approximate space-charge tune shift, where only the linear terms are retained. This seems to be a reasonable approximation as discussed in Ref. [4]. The new result is applied to the LHC at injection in the fourth section.

DISPERSION RELATION

Considering the case of a quasi-parabolic distribution function having the same normalized rms beam size $\sigma = \sqrt{\varepsilon}$ in both transverse planes and taking into account both octupoles and nonlinear space-charge forces, the Landau damping mechanism of coherent instabilities, e.g. in the horizontal plane, is discussed from the following dispersion relation [1,5]

$$1 = -\int_{J_x=0}^{+\infty} dJ_x \int_{J_y=0}^{+\infty} dJ_y \, \frac{J_x \dfrac{\partial f\left(J_x, J_y\right)}{\partial J_x}\left[\Delta Q^x_{coh} - \Delta Q^x_{incoh}\left(J_x, J_y\right)\right]}{Q_c - Q_x\left(J_x, J_y\right) - m Q_s}\,, \tag{1}$$

with, for $J_x + J_y \leq J_{\max} = 5\,\sigma^2$,

$$f\left(J_x, J_y\right) = \frac{12}{J^2_{\max}}\left(1 - \frac{J_x + J_y}{J_{\max}}\right)^2, \tag{2}$$

$$Q_x\left(J_x, J_y\right) = Q_{x0}\left(J_x, J_y\right) + \Delta Q^x_{incoh}\left(J_x, J_y\right). \tag{3}$$

Here, Q_c is the coherent betatron tune to be determined, $J_{x,y}$ the betatron action variables in the horizontal and vertical plane respectively, ΔQ^x_{coh} and ΔQ^x_{incoh} the horizontal coherent and incoherent tune shifts, m the head-tail mode number, Q_s the small-amplitude synchrotron tune (the longitudinal spread is neglected), and $Q_{x0}\,(J_x, J_y)$ the horizontal tune in the presence of octupoles but in the absence of space-charge, given by [6]

$$Q_{x0}\left(J_x, J_y\right) = Q_{x00} + a\,J_x + b\,J_y\,. \tag{4}$$

To be consistent, the space-charge tune shift has to be computed for the assumed quasi-parabolic distribution function of Eq. (2), which is done in the next section.

NONLINEAR INCOHERENT SPACE-CHARGE TUNE SHIFT

Poisson's equation can be integrated explicitly for the special case of ellipsoidal symmetry [7]. This leads to the expression of the Lorentz force experienced by the particle located at the position (x, y) inside the bunch. For an approximate solution, the nonlinear x- and y-dependence of the force is converted into an amplitude dependence of the particle's tune using the method of the harmonic balance, which is an averaging process over the incoherent betatron motions. The self-consistent nonlinear space-charge tune shift is finally given by [4]

$$\Delta Q^x_{incoh}\left(j_x, j_y\right) = \Delta_0 \begin{bmatrix} 1 - \frac{9}{8} j_x - \frac{3}{4} j_y + \frac{5}{8} j_x^2 + \frac{3}{4} j_x \; j_y \\ + \frac{3}{8} j_y^2 - \frac{35}{256} j_x^3 - \frac{15}{64} j_x^2 \; j_y \\ - \frac{27}{128} j_x \; j_y^2 - \frac{5}{64} j_y^3 \end{bmatrix}, \quad (5)$$

with

$$\Delta_0 = - \frac{N_b \, r_p}{5 \, \pi \, B \, \beta \, \gamma^2 \varepsilon^{norm}_{rms}} , \quad (6)$$

where, $j_x = J_x / J_{\max}$, $j_y = J_y / J_{\max}$, N_b is the number of protons in the bunch, r_p the classical proton radius, B the bunching factor, β and γ the relativistic velocity and mass factors, and $\varepsilon^{norm}_{rms} = \beta \gamma \varepsilon$ the transverse rms normalized emittance.

SOLUTION OF THE DISPERSION RELATION

Knowing the expression of the nonlinear space-charge tune shift, the dispersion relation of Eq. (1) can then be expressed [4]. This equation has not been solved yet. As discussed in Ref. [4], a reasonable approximation of the space-charge tune shift is given by taking into account only the linear terms in the betatron action variables $J_{x,y}$ (adapting the coefficients!). In this case, it is written

$$\Delta Q^x_{incoh}\left(J_x, J_y\right) = \Delta_0 + \Delta_a \; J_x + \Delta_b \; J_y \, . \quad (7)$$

The dispersion relation can then be solved analytically and is expressed as

$$\Delta Q^x_{coh} = \Delta_0 + \frac{1}{K_1\left(c_1, q\right)} \times \left[\frac{S_1}{24} + J_{\max} \, \Delta_a \, K_2\left(c_1, q\right) + J_{\max} \, \Delta_b \, K_3\left(c_1, q\right) \right], \quad (8)$$

with

$$K_1\left(c_1, q\right) = - \frac{1}{6 \, c_1^2 \, \left(c_1 - 1\right)^2} \times \left\{ \begin{array}{l} \left(c_1 + q\right)^3 \log\left(1 + q\right) - \left(c_1 + q\right)^3 \log\left(c_1 + q\right) \\ + \left(c_1 - 1\right) \left\{ c_1 \left[c_1 + 2 \, c_1 \, q + \left(2 c_1 - 1\right) q^2 \right] \right\} \\ + \left(c_1 - 1\right) q^2 \left(3 c_1 + q + 2 \, c_1 \, q\right) \left[\log\left(q\right) - \log\left(1 + q\right) \right] \end{array} \right\}, \quad (9)$$

$$K_2\left(c_1, q\right) = - \frac{1}{24 \, c_1^2 \, \left(c_1 - 1\right)^3} \times \left\{ \begin{array}{l} \left(c_1 - 1\right) c_1 \left\{ \begin{array}{l} c_1 - 3 c_1^2 - 2 \, c_1 \left(c_1 + 2\right) q + c_1 \left(-11 + 5 \, c_1\right) q^2 \\ + 2 \left[1 + c_1 \left(-5 + 3 c_1\right)\right] q^3 \end{array} \right\} \\ - 2 \left(c_1 + q\right)^4 \log\left(1 + q\right) \\ + 2 \left\{ \begin{array}{l} \left(c_1 + q\right)^4 \log\left(c_1 + q\right) \\ + \left(-1 + c_1\right)^3 q^3 \left(4 \, c_1 + q + 3 \, c_1 \, q\right) \left[\log\left(q\right) - \log\left(1 + q\right) \right] \end{array} \right\} \end{array} \right\}, \quad (10)$$

$$K_3\left(c_1, q\right) = - \frac{1}{24 \, c_1^3 \, \left(c_1 - 1\right)^3} \times \left\{ \begin{array}{l} \left(-1 + c_1\right) c_1 \left\{ \begin{array}{l} c_1^2 \left(1 + c_1\right) + 6 \, c_1^2 \, q + 3 \, c_1 \left(1 + c_1\right) q^2 \\ + 2 \left[1 + \left(-1 + c_1\right) c_1\right] q^3 \end{array} \right\} \\ + 2 \left(c_1 + q\right)^3 \left(c_1 - q + 2 \, c_1 \, q\right) \log\left(1 + q\right) - 2 \left(c_1 + q\right)^3 \\ \left(c_1 - q + 2 \, c_1 \, q\right) \log\left(c_1 + q\right) \\ + 2 \left(-1 + c_1\right)^3 q^3 \left[q + c_1 \left(2 + q\right)\right] \left[\log\left(q\right) - \log\left(1 + q\right) \right] \end{array} \right\}, \quad (11)$$

where $c_1 = b_1 / a_1$, $a_1 = a + \Delta_a$, $b_1 = b + \Delta_b$, $S_1 = - a_1 \, J_{\max}$ and $q = \left(Q_c - Q_{x00} - m \, Q_s - \Delta_0\right) / S_1$.

APPLICATION TO THE LHC AT INJECTION

The case of the LHC at injection is considered. Given the nominal beam emittance at 450 GeV/c $\varepsilon = 7.8$ nm, and the maximum permitted octupole spread (compatible with an adequate dynamic aperture), the corresponding values of the anharmonicities are $a \approx \pm 7164$ and $b \approx \mp 4647$. Making the numerical computation for the nominal LHC beam parameters gives $\Delta_0 \approx -1.1 \times 10^{-3}$, $\Delta_a \approx 18127$ and $\Delta_b = 12948$ [4].

Four stability diagrams are represented and compared in Fig. 1 with the nominal LHC beam parameters and maximum permitted octupolar strength: $a > 0$ corresponds to the case with octupoles only and positive horizontal detuning, $a < 0$ corresponds to the case with octupoles only and negative horizontal detuning, $a > 0 + \mathrm{SC}$ corresponds to the case with both space-charge and positive horizontal detuning for the octupoles, $a < 0 + \mathrm{SC}$ corresponds to the case with both space-charge and negative horizontal detuning for the octupoles. If ΔQ^x_{coh} (complex coherent tune shift in the absence of tune spread) lies on the inside of the stability diagram, the beam is stable. If it lies on the outside, the beam is unstable. Without frequency spread, the condition for the beam to be stable is thus simply $\mathrm{Im}(\Delta Q_{coh}^{x}) \geq 0$ (oscillations of the form $e^{j\omega t}$ are considered).

The evolution of theses four stability diagrams with decreasing space-charge and octuplar strength is shown in Figs. 2 and 3 respectively. It is seen that when space-charge (i.e. intensity) decreases, the stability diagrams converge to the ones found by Berg and Ruggiero without space charge [8]. When the octupolar strength is reduced, the stability diagrams converge to each other and to zero, as predicted by Möhl and Schönauer [1].

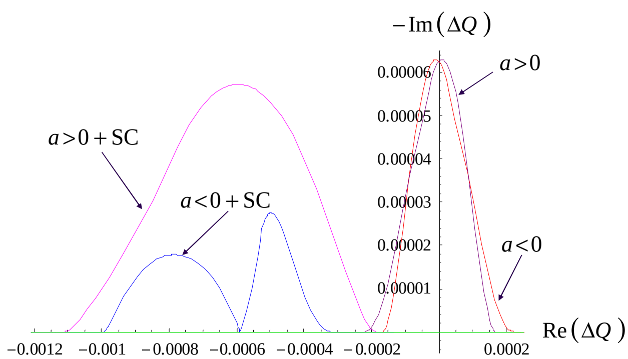

Figure 1: Stability diagrams for the nominal LHC beam parameters at injection with maximum octupolar strength.

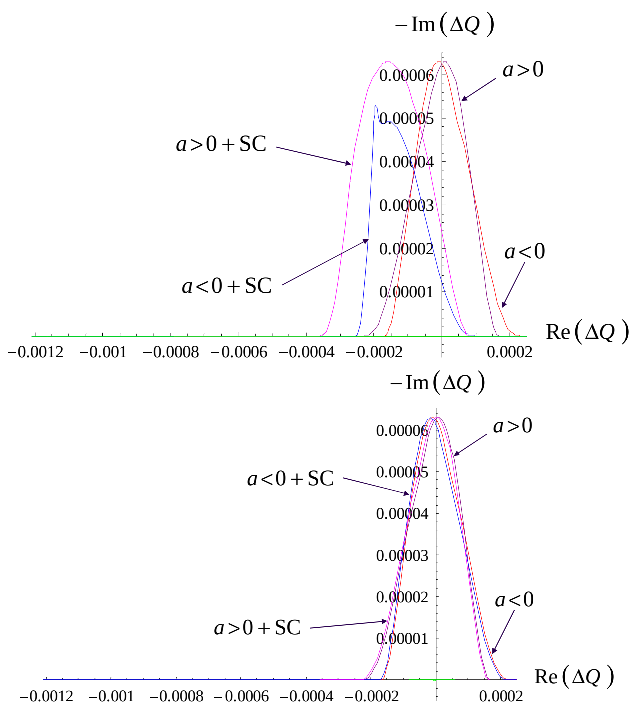

Figure 2: Evolution of the stability diagrams with decreasing space-charge (intensity): (upper) $N_b/4$ and (lower) $N_b/100$.

CONCLUSION

The self-consistent nonlinear space-charge tune shift corresponding to the quasi-parabolic distribution function

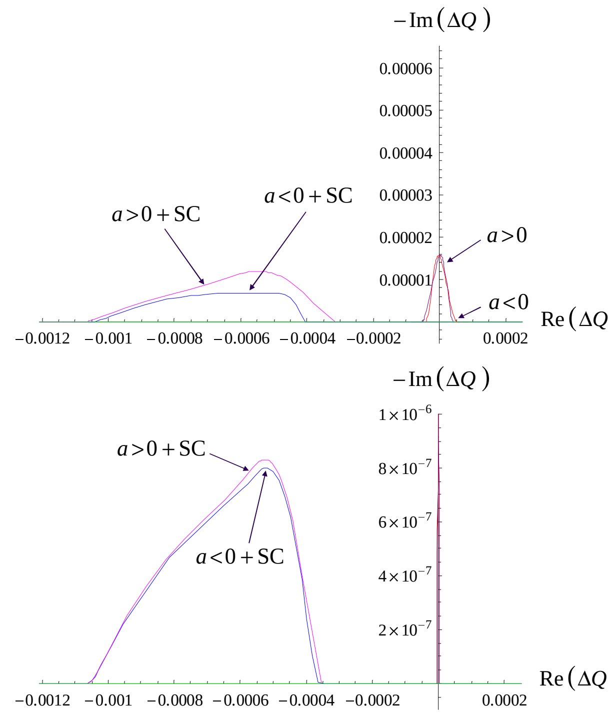

Figure 3: Evolution of the stability diagrams with decreasing octupolar strength: (upper) $1/4$ of the maximum strength and (lower) $1/50$. Note the change of vertical scale for the second plot.

$f(J_x, J_y) \propto [1-(J_x+J_y)/(5\sigma^2)]^2$ has been derived analytically. A reasonable approximation of it is provided by the linear terms in the betatron action variables. The dispersion relation has been solved analytically in this case.

The approximate tune shift should give a reasonable picture for beams with rectangular longitudinal profiles, "slightly" underestimating the effect of the large-amplitude particles, which should reflect on the shape of the stability diagram but neither on the height nor on the width.

For the usual case of parabolic or Gaussian bunches, the last missing important ingredient in this theory is the longitudinal variation of the transverse space-charge forces, which should increase the stability region on the right-hand side.

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