

# TRANSVERSE RESISTIVE WALL IMPEDANCE AND WAKE FUNCTION WITH “INDUCTIVE BYPASS”

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## Abstract

A new physical regime is revealed by the LHC collimators. Small aperture paired with large wall thickness ask for a different physical picture of the resistive wall effect than the classical one. Two general formulae for multi-layer resistive-wall impedances recently derived are reviewed and compared in the particular case of an LHC collimator. The corresponding wake function, which is useful for the simulation of beam stability in the time domain, has been obtained.

## INTRODUCTION

The first unstable betatron line in LHC is around 8 kHz, where the skin depth for graphite ( $\rho = 14 \mu\Omega m$ ) is  $\simeq 2$  cm. It is close to the collimator thickness  $d$ , which is designed to be 2.5 cm. Hence one could think that the resistive thick-wall formula would be about right. In fact it is not, as will be shown below. The resistive impedance is about 2 orders of magnitude lower at this frequency.

Concerning the LHC collimators and their effect on beam stability, it is found here that standard transverse resistive-wall impedance  $Z_{m=1}^\perp$  [1], both thick and thin wall,

$$Z_{m=1}^{\perp, \text{thick}}(\omega) = (\text{sgn } \omega + j) \frac{Z_0 L \delta_0 \mu_r}{2 \pi a^3} \cdot \sqrt{\frac{\omega_0}{|\omega|}}, \quad (1)$$

$$Z_{m=1}^{\perp, \text{thin}}(\omega) = \frac{cL}{\pi a^3 \sigma_c d \cdot \omega}, \quad (2)$$

would give too pessimistic results, especially for low frequencies. Definition of symbols can be found in eg. [2]. Any expression is given in MKSA units and the convention  $\exp j \omega t$  for forward time propagation is used.

Different resistive-wall formulae have been recently developed [3, 4, 5] using distinct approaches in their derivation. Corresponding measurements [6] are available for standard beam pipe dimensions, where the wall thickness  $d$  is small compared to the beam pipe radius  $a$ . In this case all below presented formulae give good agreement. The LHC collimators represent an extreme case, where the distance to the beam is small compared to the collimator thickness.

The general multi-layer resistive wall impedances [3, 4, 5], although derived by completely different methods, show some similarities and complement earlier results [7, 8]. An attempt to establish a corresponding physical picture is undertaken.

## GENERAL MULTI-LAYER FORMULA BY CIRCUIT ANALYSIS

L.Vos (LV) gave a procedure to obtain the transverse impedance of a cylindrical beam pipe from an arbitrary longitudinal surface impedance [4] by including a so-called

“inductive bypass”, also referred to as “redistribution of the image current”. Once the longitudinal impedance  $Z_{\parallel}$  per unit length is known the transverse impedance  $Z_{\perp}^{\text{ibp}}$  per unit length can be computed from  $Z_{\perp}^{\text{ibp}} = 2c / (a^2 \omega) \cdot Z_{\parallel} Z_{\text{ind}} / (Z_{\parallel} + Z_{\text{ind}})$ , where the inductive bypass is given by  $Z_{\text{ind}} = j \omega \mu_0 / 4\pi$ . Historically, applying this concept to (1) was used to get a more realistic impedance estimate for the LHC collimators (see also (5)).

Additionally LV proposed to compute the longitudinal impedance of multi-layer vacuum chambers using the theory of transmission lines, which implicitly takes into account Maxwell’s equations with the proper boundary conditions [5].

Both formalisms combined give transverse impedance expressions for any number of layers with any electric conductivity  $\sigma$ , magnetic permeability  $\mu$  and dielectric constant  $\epsilon$ . The formalism for  $n$  layers assumes the outermost layer to extend to infinity, i.e. it loads the previous layer with its intrinsic impedance  $Z_n$ . Using for layer  $m$  its intrinsic impedance  $Z_m = \sqrt{j \omega \mu_m / (j \omega \epsilon_m + \sigma_m)}$ , its radial propagation constant  $\gamma_m = \sqrt{j \omega \mu_m (\sigma_m + j \omega \epsilon_m) + (\omega / \beta c)^2}$  and its thickness  $d_m$ , the input impedance  $Z_{i,m}$  per unit length of each layer can be computed recursively

$$Z_{i,m} = Z_m \frac{Z_{i,m+1} \cosh \gamma_m d_m + Z_m \sinh \gamma_m d_m}{Z_m \cosh \gamma_m d_m + Z_{i,m+1} \sinh \gamma_m d_m}. \quad (3)$$

Finally the longitudinal impedance per unit length  $Z_{\parallel} = Z_{i,1} / (2\pi a)$  is used to compute the transverse impedance as described before.

The approach requires that the beam’s wave length is large compared to the beam pipe radius  $\beta c / \omega \gg a$ . This is almost always satisfied in any accelerator. However, further limits of applicability are not specified in references [4, 5].

The case of a finite chamber wall ( $n = 2$ , where layer 2 is vacuum extending to infinity) gives

$$Z_{m=1}^{\perp, \text{LV}}(\omega) = \frac{Z_0 L}{2 \pi a^2} \cdot \left[ \frac{a \omega \mu_0}{2 Z_0} \cdot \frac{1 + \frac{Z_0}{Z_1} \tanh \gamma_1 d_1}{1 + \frac{Z_1}{Z_0} \tanh \gamma_1 d_1} - j \right]^{-1}. \quad (4)$$

Taking the limit  $|\gamma_1 d_1| \gg 1$  in (4) gives

$$Z_{m=1}^{\perp, \text{thick,ibp}}(\omega) = (1 + j \text{sgn } \omega) \frac{Z_0 L}{2 \pi a^2} \cdot \frac{1}{-j + \text{sgn } \omega \left( 1 + a \sqrt{\frac{\sigma_c \mu_0}{2 \mu_r}} \sqrt{|\omega|} \right)}, \quad (5)$$

which can be obtained as well by applying the concept of inductive bypass to (1). Expression (5) was originally used for the impedance estimates of LHC collimators. Note that the assumption  $d \rightarrow \infty$  does not entirely reflect the physics we want to model, however the formula gives good results for the parameters under consideration. Applying the concept of inductive bypass to (2) or on the other hand taking the limit  $|\gamma_1 d_1| \ll 1$  in (4) gives

$$Z_{m=1}^{\perp, \text{thin,ibp}}(\omega) = \frac{Z_0 L}{2\pi a^2} \cdot \frac{1}{\frac{1}{2} a d \sigma \mu_0 \cdot \omega - j}, \quad (6)$$

which is a known formula for thin metal walls already reported in [9].

## RESISTIVE WALL WAKE FUNCTION WITH INDUCTIVE BYPASS

The corresponding wake function of (5) has been found [2] and is given by

$$W_{m=1}^{\perp}(t > 0) = +\frac{cL}{\pi^{3/2} a^3} \sqrt{\frac{\mu_o \mu_r}{\sigma_c}} \cdot \frac{1}{\sqrt{|t|}} - \exp\left[\frac{4\mu_r}{a^2 \sigma_c \mu_0} |t|\right] \frac{2cL\mu_r}{a^4 \pi \sigma_c} \cdot \left(1 - \text{Erf} \sqrt{\frac{4\mu_r}{a^2 \sigma_c \mu_0} |t|}\right). \quad (7)$$

It was used in simulation studies for the LHC and SPS. The wake function (7) reveals the classical result in the first term and adds an additional corrective term which gives its dominant contribution in the intermediate range as it vanishes in the limit  $t \rightarrow \infty$  and approaches a finite value in the limit  $t \rightarrow 0$ .

Referring to the discussion subsequent to (5) the wake function gives good results for the present collimator parameters and for the standard parameter range ( $d < a$ ).

## GENERAL MULTI-LAYER FORMULA USING A QUASI-STATIC BEAM MODEL

In [3] Burov and Lebedev (BL) follow a different approach to calculate the transverse resistive wall impedance. They use a quasi-static beam model, where the beam is represented by an electric plus a magnetic dipole. The solution is then accomplished by solving Poisson's equation for the electric dipole and computing the longitudinal vector potential associated with the magnetic dipole.

The technique requires that the beam's wave length is large compared to the beam pipe radius  $\beta c/\omega \gg a$ . Furthermore the assumption  $L \gg a$  is needed, hence the structure should be long compared to the aperture. This is in particular the case for the LHC collimators, where  $L/a \approx 500$ .

The formalism allows also analytic impedance calculation for any number of layers. By using the wave vector  $\kappa_m = \sqrt{j \sigma_m \mu_r \mu_0 \omega}$  and the medium parameter  $\tilde{\kappa}_m = \kappa_m / \mu_r$  associated with layer  $m$ , the transfer from a layer  $m$  to a layer  $m-1$ , with  $2 \leq m \leq n-1$ , is found

as

$$T_{m-1} = -\frac{c'_{m-1}(\kappa_{m-1} a_m) - T_m \tilde{\kappa}_{m,m-1} c_{m-1}(\kappa_{m-1} a_m)}{s'_{m-1}(\kappa_{m-1} a_m) - T_m \tilde{\kappa}_{m,m-1} s_{m-1}(\kappa_{m-1} a_m)}, \quad (8)$$

where  $s$  and  $c$ , the basis sine and cosine hyperbolic Bessel solutions and their derivatives are defined in [3]. Starting the recursion scheme with  $T_n = 1$ , letting  $\tilde{\kappa}_{m,m-1} = \tilde{\kappa}_m / \tilde{\kappa}_{m+1}$  and  $\tilde{\kappa}_n = 1/(a_n \mu_r)$  for a non-conducting unbounded outermost layer, the impedance is finally given by

$$Z_{\perp}(\omega) = j \frac{Z_0 \beta L}{2\pi a^2} \cdot \frac{2}{1 - \tilde{\kappa}_1 a T_1}. \quad (9)$$

The particular case of only two layers ( $n = 2$ , where layer 2 is vacuum extending to infinity) gives the general expression for the transverse resistive-wall impedance,

$$Z_{m=1}^{\perp, \text{BL}}(\omega) = j \frac{Z_0 \beta L}{\pi a^2} \frac{s'_1 + \tilde{\kappa}_{2,1} s_1}{s'_1 + \tilde{\kappa}_{2,1} \tilde{\kappa}_{1,0} c_1 + \tilde{\kappa}_{2,1} s_1 + \tilde{\kappa}_{1,0} c'_1}. \quad (10)$$

## COMPARISON AND DISCUSSION

The heuristic inclusion of the "inductive bypass" comes remarkably close to the result by BL as can be seen qualitatively in Fig.1. This is why stability estimates established with (5) still hold.

However, the LV approach has limitations that are inherent in transmission line theory. Hyperbolic tangents appear, which obviously refer to further assumptions that are not specified. This becomes apparent for the copper coated LHC collimator, where the LV result significantly differs from the BL impedance (see Fig.2). The BL approach is more rigid from the beginning and the limits of applicability are specified.

As can be seen from Figs. 1 and 2, the real part of the transverse impedance is largely reduced. Usually the condition  $\delta_{\text{skin}} \leq a$  holds for all frequencies in consideration. The lower the frequency the larger the skin depth. Pair this with small aperture and we easily arrive at the condition  $\delta_{\text{skin}} \gg a$ , as in the case of the LHC collimators. Then the beam does not see the induced currents at the distance  $a$  any more, but instead sees them over the range  $[a, \delta_{\text{skin}}]$ , thus the effective aperture is in the order of  $\delta_{\text{skin}} \gg a$  in this case [10].

The imaginary part at low frequencies is dominantly determined by the effect of the electric dipole. It is simply the image of the beam on the surface of the vacuum chamber. The surface is at fixed distance  $a$ , thus  $\text{Im } Z$  approaches a constant value for  $\omega \rightarrow 0$ . The magnetic dipole does not contribute at low frequencies, because the related electric field goes to zero,  $j_z = \sigma E_z = \partial A_z / \partial t \propto \omega \rightarrow 0$ ; i.e. the chamber does not respond to the magnetic dipole.

Fig. 1 shows the impedance of an uncoated LHC collimator. At 8 kHz the real part of the impedance is reduced by a factor 100, which is consistent with the picture of effective aperture just explained.

Results for a copper coated LHC collimator, with  $5 \mu\text{m}$  coating thickness, are portrayed in Fig. 2. Pay attention to

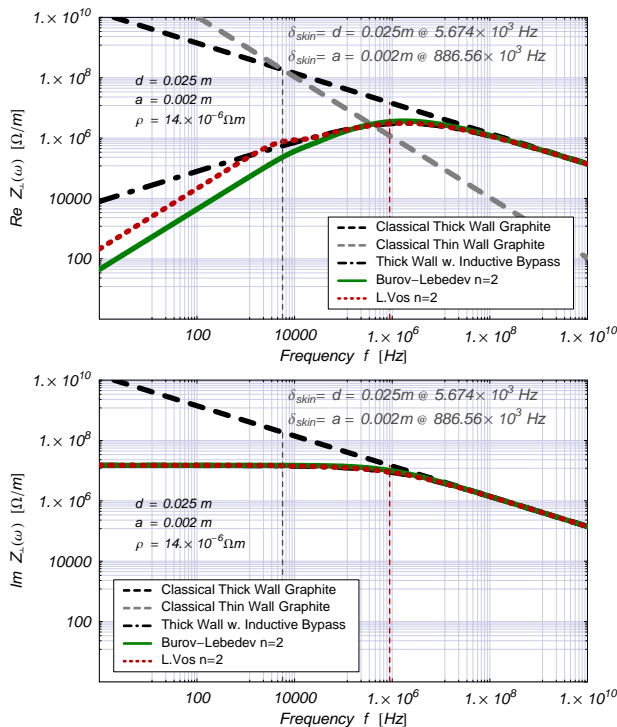


Figure 1: Real and imaginary part of different formulae for the transverse resistive wall impedance for the case of an LHC collimator,  $L = 1$  m.

the fact that in this case the real part of the impedance is increased below 1 MHz and reduced above as compared to the uncoated example. At high frequencies the induced current is more and more confined to the copper layer, hence the impedance is reduced. At low frequencies similarly a significant part of the current will flow in the copper layer with less resistivity. Then the effective aperture is reduced, thus the real part of the impedance increases accordingly. These results are consistent with HFSS simulations [11].

Note that each derivation presented results in only one formula from zero frequency up to  $\omega \ll \beta c/a$ , without the need to distinguish between cases where the skin depth is smaller or bigger than the wall thickness. In particular the characteristics of all presented impedances converge for the “standard” parameter range ( $d < a$ ) (see [6]).

### CONCLUSION AND OUTLOOK

We have shown that for low frequencies and wall thicknesses larger than the vacuum chamber radius ( $d > a$ ), the behavior of the transverse resistive-wall impedance deviates significantly from common understanding. A physical picture is established that explains the changes qualitatively. The analyzed impedances and wake function are in particular reasonable for the LHC collimator parameters and moreover generally valid for usual parameters (i.e.  $d < a, \delta_{skin} < a, L \gg a$ )

As a general remark, we retain that the results presented in this paper are obtained assuming 2D models, which

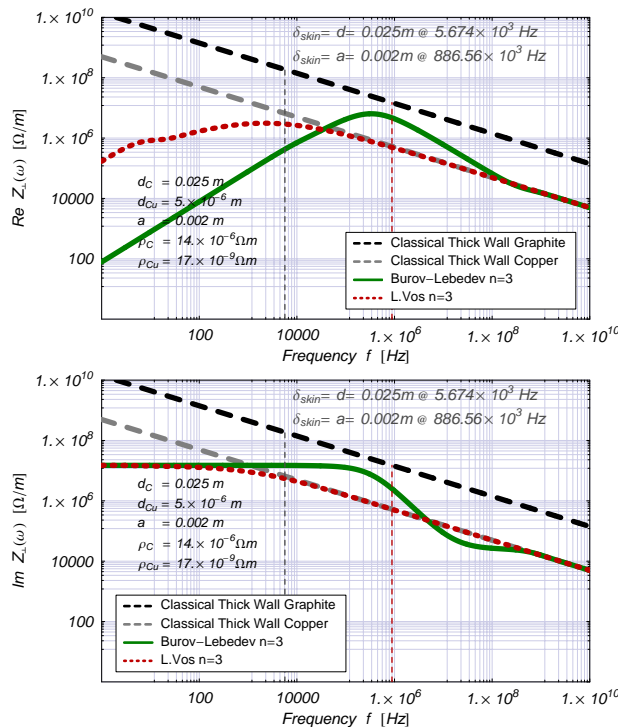


Figure 2: Real and imaginary part of the transverse resistive wall impedance with copper coating ( $5 \mu\text{m}$ ) for the case of an LHC collimator,  $L = 1$  m.

appears justified for the parameters under consideration ( $L \gg a$ ). A measurement campaign is planned for this year with an LHC collimator prototype, both with beam in the CERN SPS and as a bench measurement, using a vibrating wire method which is believed to be superior in sensitivity and applicability compared to the two-wire method for the geometry under consideration. For structure lengths smaller or comparable to the transverse dimensions, the 2D model may be no longer applicable [12]. The limits of applicability still have to be explored in more detail.

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