

BEAM INVARIANTS FOR DIAGNOSTICS

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Abstract

This paper describes measurable quantities of beams preserved under symplectic transformations. General beam distributions have no determined area, and rms quantities of the beam do not provide invariants in the general nonlinear case. Here we show that in the 1D case there exist integral and local invariants, directly linked to Liouville's theorem. Beam invariants, related to general properties of symplectic transformations, are also found and presented for 2D and 3D cases. If measured at different locations, these invariants can tell whether the transformation is symplectic, or there exist diffusion, friction, or other non-Hamiltonian dynamic processes in the beam.

INTRODUCTION

The Hamiltonian dynamics and associated symplectic transformations are a very well developed subject. For diagnostics purposes, however, all the results from symplectic geometry are hardly applicable. The reason is that the beam consists typically of identical particles and it is very hard to know which part of the beam is transformed into any particular spot, especially when the dynamics are governed by a space charge force that depends on the measured distribution. All we have is projections of the beam distribution at various points. Under typical circumstances many initial conditions will produce identical measured beam profiles. For example, in LEDA beam experiments [1], intense beam profiles were measured at several locations and there was a possibility to find initial distributions that reproduce the measured beam profiles for various sets of quadrupoles. The question is: Are there any beam characteristics, which don't depend on initial conditions? It turns out that if the dynamics of particles are determined by external and space charge forces, it is possible to find distribution properties which are preserved during beam transport. What first comes to mind is the Liouville's theorem of phase volume preservation, but fact is that for arbitrary distributions there is no such thing as volume. The rms quantities can be used for invariants only when the transport is linear (the invariants of linear motion for rms quantities were obtained in [2]). Regardless, for some smooth distributions there exist possibility to determine whether two measured distributions are related by a symplectic transformation. We present full solution to the problem for the 1D case in Section II. Solution for the beam core to the 2D and 3D cases is presented in Section III, and a practical device to measure the invariants in Section IV. Conclusion summarizes the results.

1D INVARIANTS

We begin with 1D motion and its 2D phase space. Figure 1 presents two different distributions (left and right on top) and their topological views. How can we tell if these two distributions are symplectically equivalent (i.e., can be transformed one to another by a symplectic map)? Let's take the lines in topological graphs, along which the particle's density is constant. Because the phase space density behaves as an incompressible liquid, a line of constant density is transformed by a symplectic map to a line of constant density as well. Here is the necessary condition: two correspondent (i.e. closed lines with same number of particles within the encompassed area) constant density lines of two symplectically equivalent distributions must have the same encompassed phase space area. Intuitively, this is also a sufficient condition, i.e., if the phase area of the two 2D figures is the same, there exists a symplectic map, transforming one figure to another.

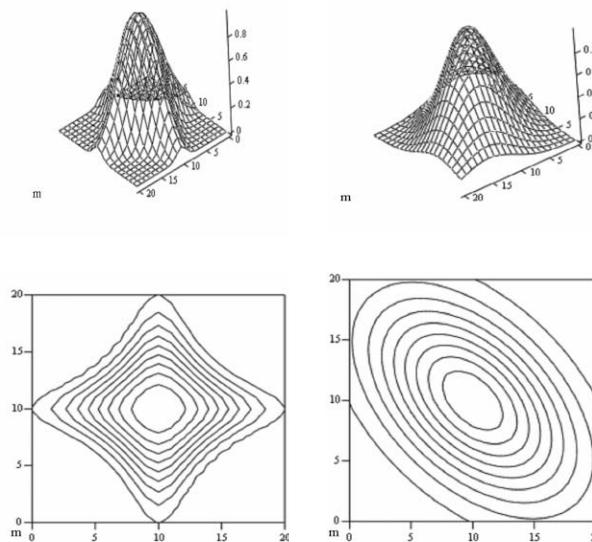


Figure 1: Sample phase space distributions of initial and final beam. Upper plots show the distributions, lower plots show their topological views. The areas encompassed by equivalent density lines must be the same.

The situation for higher dimensions can be analyzed similarly, but the conditions become more involved. Instead of going into the topology, we give simple examples for the beam core near the maximum of

distributions, assuming the function is smooth and parabolic in the vicinity of its maximum.

INVARIANTS FOR THE BEAM CORE

Now, if we look at the distributions in Figure 1, we see elliptical density lines near the maxima. This is typical for the majority of beam distributions – they have one maximum and parabolic density in the nearby vicinity. Below we consider smooth distributions with one maximum. The Taylor expansion about the maximum has only quadratic and higher terms (we present the case for any number of dimensions):

$$f(x, x', y, \dots) = ax^2 + 2bx'x + 2cxy \dots = X^T Q X, \quad (1)$$

where $f(x, x', y, \dots)$ is the distribution function, $X = [x, x', y, \dots]$, and Q is the matrix of coefficients.

For example, for the 2D case the matrix Q is:

$$Q = \begin{bmatrix} a & b & c & d \\ b & e & f & g \\ c & f & h & i \\ d & g & i & j \end{bmatrix}. \quad (2)$$

Because the maximum of one distribution is transformed into maximum again, we can linearize the symplectic transformation around this point. The vector X of small deviations from the maximum point is transformed to another vector $X' = MX$ by symplectic matrix M . The condition of symplecticity is $M^T S M = S$, where S is:

$$S = \begin{bmatrix} 0 & 1 & \dots \\ -1 & 0 & \dots \\ \dots & \dots & \dots \end{bmatrix}. \quad (3)$$

Let's find how the form Q is transformed. For this, we substitute new $X' = MX$ into the distribution expansion near the maximum:

$$f(X') \approx X'^T Q' X' = X^T M^T Q' M X, \quad (4)$$

where Q' is the matrix of quadratic coefficients after the transformation. Because the distribution expression in the old coordinates gives us the initial distribution, we have:

$$M^T Q' M = Q. \quad (5)$$

Multiplying (5) by the S matrix, and using $S^2 = -E$, where E is the unity matrix, and $M^T S M = S$, we yield:

$$-S M^T S M M^{-1} S Q' M = M^{-1} S Q' M = S Q. \quad (6)$$

This means that the matrices SQ' and SQ are *similar* and that their eigenvalues are equal and are invariants of the motion. Therefore, in order to check the symplectic equivalency of two beams near their maxima, we have to find the matrices Q of the quadratic forms, multiply by the matrix S , and find their eigenvalues. If they are equal, then there exists a *similarity* transformation of the type (5) between them. If the matrix of this transformation is

symplectic, this matrix itself represents the transformation of distribution near its maximum. Now we have to count the number of independent invariants. To do this, we consider the equation for the eigenvalues λ of matrix the SQ :

$$|SQ - \lambda E| = 0. \quad (7)$$

First, let's transpose matrices in (7). Using the fact that Q is symmetric (i.e. $Q^T = Q$), $S^2 = -E$, and $S^T S = E$ we have:

$$\begin{aligned} |SQ - \lambda E| &= |QS^T - \lambda E| = |S^{-1}QS^T S - \lambda E| = \\ &= |-SQ - \lambda E| = |SQ + \lambda E| = 0. \end{aligned} \quad (8)$$

Here we used the fact that the transposed matrices have same eigenvalues. The second equality of (8) is a consequence of the fact that *similar* matrices (related to each other by a transformation of type (5)) have the same eigenvalues. The third equality comes from the relation $-S^T = S$. Equation (8) shows that if the matrix SQ has an eigenvalue λ , $-\lambda$ is also an eigenvalue. Therefore, the polynomial (7) has only even orders of λ . All the coefficients of the various even orders of λ in (7) are invariants. The first coefficient of the term λ^{2n} is always equal to one; the others are nontrivial functions of matrix elements of Q . For an arbitrary dimension n , the matrix Q is $2n$ dimensional, and the number of nontrivial invariants is equal to n . For 2D motion quadratic form (2) has two invariants:

$$\begin{aligned} ae - b^2 + jh - i^2 + 2(gc - fd) &= inv, \\ \det(Q) &= inv. \end{aligned} \quad (9)$$

SCHEMES FOR DIAGNOSTICS

Measuring the invariants of motion introduced in the previous Section can provide a test for the Hamiltonian nature of beam motion between two locations. The most straightforward way to calculate the invariant (9) is to measure the distribution function $f(x, x', y, \dots)$ then to obtain the matrix Q by expanding f in a Taylor series about the maximum. Direct measurement of a distribution function is not in the mainstream of beam diagnostic tasks. Measurement of various projections of full 4D or 6D phase space of the beam is what is usually provided. Examples include vertical, horizontal, and longitudinal profiles (1D projections); vertical, horizontal and longitudinal "emittances" (2D projections). 2D projection can be restored from the 1D profiles using tomographic techniques [3], but no other similar methods have been developed yet, to our knowledge, for restoring 4D or 6D distribution from 2D projections.

Measuring the distribution function for 2-D beams (4D phase space) should not pose a significant problem. One example of a device suitable for that is the so-called "pepper pot", which is common in beam diagnostics. A schematic view of the "pepper pot" is shown in Fig.2. A screen (1) with pin holes (2) is used for spatial coordinates (x, y) selection. A detector screen (2)

locates angle coordinates (x', y') . Beam current passing through (x, y) and detected at (x', y') is proportional to $f(x, x', y, y')$. In order to avoid overlapping of the images of individual pinholes on the second screen, pin holes on the first screen must be farther from each other than the distance defined by the angular spread in the beam. As a result, a “pepper pot” doesn’t measure a continuous distribution function, but provides discrete samples with some unavoidable granularity. This intrinsic feature of the “pepper pot” device doesn’t affect the ability to calculate the matrix Q and the invariants because we need to find only a finite number of quadratic coefficients of the Taylor expansion (10 for 4D distribution), which can be done using a finite number of samples. Another important feature of the proposed measurement is that we are only interested in measuring the beam density near its maximum, where noise problems are less significant than in the tails.

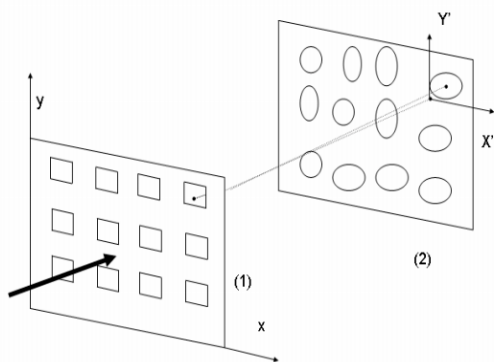


Figure 2: Schematic view of 4D beam distribution function measurements using “pepper pot” device. (1) Screen with pin holes. (2) Detector screen

In case of 3D beams, a generalized variant of the pepper pot can be constructed as depicted in Fig. 3. Screen (1) with pin holes (2) is used for the spatial coordinates (x, y) selection again. Screen (2) now has a single movable pin hole for the (x', y') selection. A magnetic spectrometer (3) and time resolving detector (4) are used to define the energy and time (z, t) pair of coordinates. Again, we are only interested in measurements near the maximum of the beam density but noise problem can be much more severe for the 6D phase space due to the onset of the notorious “curse of dimensionality”. It is illustrated by the following simple estimate of the number of particles in the beam required for a meaningful measurement. If we measure the beam

density on a grid of 10 points per each dimension, we have 10^6 cells in the 6D space. For reliable statistics, 100 particles per cell at least are needed or 108 particles in the beam core and $\sim 10^9$ particles total in the beam.

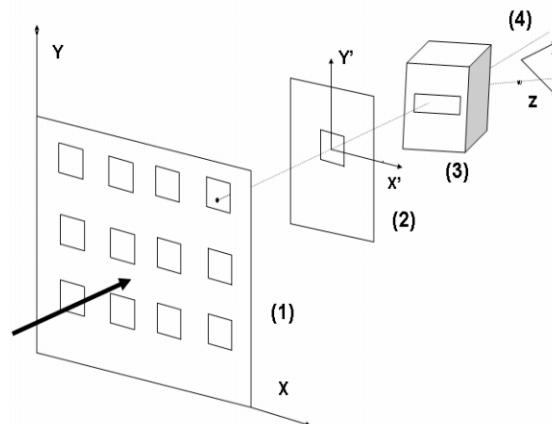


Figure 3: Schematic view of 6D beam distribution function measurements using “pepper pot” - like device. (1) Screen with pin holes. (2) Movable screen with pin hole. (3) Spectrometer. (4) Time resolving beam current detector.

CONCLUSION

This paper shows that there exist nontrivial beam invariants which are preserved if the particle’s motion is Hamiltonian in nature. The methods of retrieving full information about the beam, as well as the invariant values, are discussed.

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