

# NORMAL FORM ANALYSIS OF LINEAR BEAM DYNAMICS IN A COUPLED STORAGE RING\*

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## Abstract

The techniques of normal form analysis, well known in the literature, can be used to provide a straightforward characterization of linear betatron dynamics in a coupled lattice. Here, we consider both the beam distribution and the betatron oscillations in a storage ring. We find that the beta functions for uncoupled motion generalize in a simple way to the coupled case. Defined in the way that we propose, the beta functions remain well behaved (positive and finite) under all circumstances, and have essentially the same physical significance for the beam size and betatron oscillation amplitude as in the uncoupled case. Application of this analysis to the online modeling of the PEP-II rings is also discussed.

## INTRODUCTION

Optimal performance of electron storage rings in synchrotron light sources and circular colliders often depends on good control of the betatron coupling. Having a convenient method of characterizing the coupling becomes particularly important when the lattice includes regions where the beam is significantly coupled by design, as in the solenoid field of the interaction region of a collider. In this note, we propose a simple way to characterize the coupling. In particular, we extract from the linear single-turn map lattice functions that generalize the usual Twiss parameters for an uncoupled lattice. Our procedure avoids the ambiguities that can arise in other treatments, with associating the lattice functions with the different betatron modes [1]. This approach also has the advantage that the beta functions are directly related to the equilibrium beam distribution (for given normal mode emittances).

Our treatment is based on the normal form analysis that is well known in the beam dynamics literature [2]. The linear single-turn map at any point in a storage ring can be written as a 4x4 matrix. Normal form analysis identifies a transformation that puts this matrix into block diagonal form, with each block a simple rotation. The essential characteristics of the dynamics in the lattice are then contained in the normalizing transformation: in the uncoupled case, the components of the normalizing transformation are constructed from the usual Twiss parameters. By generalizing in a natural way to the coupled case, we extract coupled Twiss parameters with the same physical significance as those in the uncoupled case.

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In this paper, we begin by considering the betatron trajectory of a particle in a coupled lattice. We show how to construct the normalizing transformation, and how to extract the coupled beta functions from this transformation. We then relate the coupled beta functions to the equilibrium beam distribution in a storage ring, assuming Gaussian beam distribution and given beam emittance. Further details and discussion on these issues may be found in [3]. Finally, we consider application of these techniques to the online model of the PEP-II rings.

## BETATRON TRAJECTORY

Let us write the transverse phase-space vector of a particle in a lattice as:

$$|\mathbf{x}\rangle = \begin{pmatrix} x \\ p_x \\ y \\ p_y \end{pmatrix} \quad \langle \mathbf{x} | = (x \quad p_x \quad y \quad p_y)$$

We define a vector  $|\mathbf{J}\rangle$  in terms of the action-angle variables:

$$|\mathbf{J}\rangle = \begin{pmatrix} \sqrt{2J_I} \cos(\varphi_I) \\ -\sqrt{2J_I} \sin(\varphi_I) \\ \sqrt{2J_{II}} \cos(\varphi_{II}) \\ -\sqrt{2J_{II}} \sin(\varphi_{II}) \end{pmatrix}$$

The invariant action  $J_I$  is associated with the tune  $\nu_I$ ; i.e. in one turn of the lattice, the change in the phase angle  $\varphi_I$  is  $\Delta\varphi_I = 2\pi\nu_I$ . Similarly, the action  $J_{II}$  is associated with the tune  $\nu_{II}$ . Note that the fractional parts of the tunes can be found from the eigenvalues  $\lambda_I$  and  $\lambda_{II}$  of the single-turn matrix:

$$\lambda_I = \exp(\pm 2\pi i \nu_I) \quad \lambda_{II} = \exp(\pm 2\pi i \nu_{II})$$

The vectors  $|\mathbf{x}\rangle$  and  $|\mathbf{J}\rangle$  are related by a matrix  $\mathbf{N}$ :

$$|\mathbf{x}\rangle = \mathbf{N}|\mathbf{J}\rangle \tag{1}$$

$\mathbf{N}$  is readily constructed as follows. Let  $|\mathbf{e}_i\rangle$  be the eigenvectors of the single-turn matrix  $\mathbf{M}$ . The index  $i$  runs from 1 to 4 for the four eigenvectors. We order the eigenvectors so that  $|\mathbf{e}_1\rangle$  and  $|\mathbf{e}_2\rangle$  are associated with the eigenvalues  $\lambda_I$ , and  $|\mathbf{e}_3\rangle$  and  $|\mathbf{e}_4\rangle$  are associated with the eigenvalues  $\lambda_{II}$ . We also use the normalization:

$$\langle \mathbf{e}_i | \mathbf{S} | \mathbf{e}_j \rangle = \begin{cases} \pm i & \lambda_i \lambda_j = 1 \\ 0 & \lambda_i \lambda_j \neq 1 \end{cases}$$

where  $\mathbf{S}$  is the antisymmetric matrix:

$$\mathbf{S} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

Then we have:

$$\mathbf{N} = \frac{1}{\sqrt{2}} \left( |\mathbf{e}_1\rangle + |\mathbf{e}_2\rangle \frac{|\mathbf{e}_1\rangle - |\mathbf{e}_2\rangle}{i} + |\mathbf{e}_3\rangle + |\mathbf{e}_4\rangle \frac{|\mathbf{e}_3\rangle - |\mathbf{e}_4\rangle}{i} \right)$$

Since  $\mathbf{M}$  is symplectic, the eigenvectors come in complex conjugate pairs, and  $\mathbf{N}$  is real.  $\mathbf{N}$  provides a transformation that puts the single-turn matrix into block-diagonal rotation form:

$$\mathbf{N}^{-1} \mathbf{M} \mathbf{N} = \mathbf{R}(\mu_I, \mu_{II}) = \begin{pmatrix} \cos(\mu_I) & \sin(\mu_I) & 0 & 0 \\ -\sin(\mu_I) & \cos(\mu_I) & 0 & 0 \\ 0 & 0 & \cos(\mu_{II}) & \sin(\mu_{II}) \\ 0 & 0 & -\sin(\mu_{II}) & \cos(\mu_{II}) \end{pmatrix}$$

where  $\mu_{I,II} = 2\pi\nu_{I,II}$ . The relationship (1) between  $|\mathbf{x}\rangle$  and  $|\mathbf{J}\rangle$  is seen from the transformation of  $|\mathbf{x}\rangle$  under a single turn through the lattice:

$$|\mathbf{x}\rangle \rightarrow \mathbf{M}|\mathbf{x}\rangle = \mathbf{N}\mathbf{R}\mathbf{N}^{-1}|\mathbf{x}\rangle$$

from which it follows that:

$$|\mathbf{J}\rangle \rightarrow \mathbf{R}|\mathbf{J}\rangle$$

as expected.

We note that there is a degeneracy in the normalization, in that if the matrix  $\mathbf{N}$  is a valid normalization, then so is  $\mathbf{N} \cdot \mathbf{R}(\theta_1, \theta_2)$  for any angles  $\theta_1$  and  $\theta_2$ . For convenience, let us choose these angles so that the matrix elements  $n_{12}$  and  $n_{34}$  of  $\mathbf{N}$  are zero. We then *define*  $\beta_x$ ,  $\beta_y$  and quantities  $\zeta_x$ ,  $\zeta_y$  by:

$$\begin{aligned} \sqrt{\beta_x} &= n_{11} & \sqrt{\beta_y} &= n_{33} \\ \zeta_x &= n_{31} + in_{32} & \zeta_y &= n_{13} + in_{14} \end{aligned} \quad (2)$$

We then have expressions to describe the betatron motion such as:

$$x = \sqrt{2\beta_x} J_I \cos(\varphi_I) + \sqrt{2J_{II}} \operatorname{Re}(\zeta_y e^{-i\varphi_{II}})$$

This reduces to the familiar expression for  $x$  if *either*:

- There is excitation of betatron motion associated only with the tune  $\nu_I$ , i.e.  $J_{II} = 0$ , or
- The coupling parameter  $\zeta_y = 0$ , i.e. the local coupling in the lattice is zero.

## BEAM DISTRIBUTION

We need do very little extra work to find expressions for the second-order moments of the beam distribution. In fact, if we define the emittance as the average of the action of all particles in the beam:

$$\varepsilon_I = \langle J_I \rangle \quad \varepsilon_{II} = \langle J_{II} \rangle$$

then it follows immediately from equations (1) and (2) that:

$$\begin{aligned} \langle x^2 \rangle &= \beta_x \varepsilon_I + |\zeta_y|^2 \varepsilon_{II} \\ \langle y^2 \rangle &= \beta_y \varepsilon_{II} + |\zeta_x|^2 \varepsilon_I \\ \langle xy \rangle &= \sqrt{\beta_x} \operatorname{Re}(\zeta_x) \varepsilon_I + \sqrt{\beta_y} \operatorname{Re}(\zeta_y) \varepsilon_{II} \end{aligned} \quad (3)$$

Furthermore, if we define:

$$\begin{aligned} \alpha_x &= -n_{11}n_{21} & \alpha_y &= -n_{33}n_{43} \\ \tilde{\zeta}_x &= n_{23} - in_{24} & \tilde{\zeta}_y &= n_{41} - in_{42} \end{aligned}$$

then we have:

$$\begin{aligned} \langle xp_x \rangle &= -\alpha_x \varepsilon_I + \operatorname{Re}(\tilde{\zeta}_x \zeta_y) \varepsilon_{II} \\ \langle yp_y \rangle &= -\alpha_y \varepsilon_{II} + \operatorname{Re}(\tilde{\zeta}_y \zeta_x) \varepsilon_I \end{aligned}$$

Clearly, all second order moments of the beam distribution are contained in the components of the normalizing matrix  $\mathbf{N}$ . In fact, if we define matrices:

$$\mathbf{A}_I = (\mathbf{N}^{-1})^T \mathbf{C}_I \mathbf{N}^{-1} \quad \mathbf{A}_{II} = (\mathbf{N}^{-1})^T \mathbf{C}_{II} \mathbf{N}^{-1}$$

where:

$$\mathbf{C}_I = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \mathbf{C}_{II} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

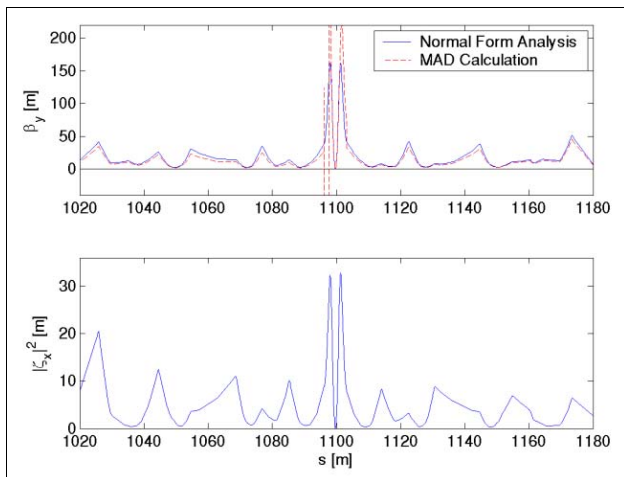
then it may be shown [3] that the matrix of the second-order beam distribution may be written:

$$\Sigma = \left( \frac{\mathbf{A}_I}{\varepsilon_I} + \frac{\mathbf{A}_{II}}{\varepsilon_{II}} \right)^{-1}$$

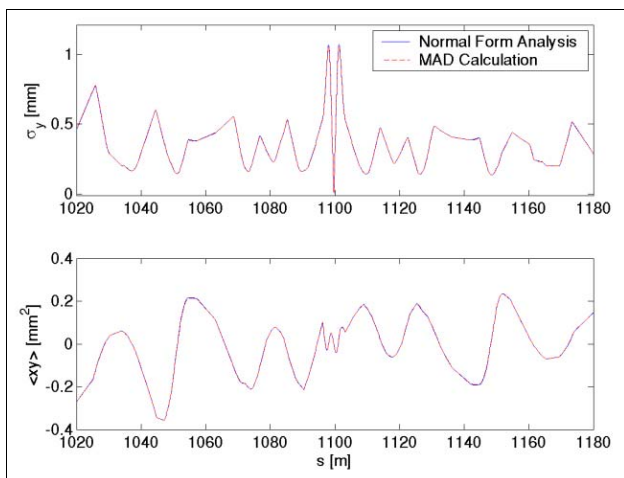
We note that these expressions generalize easily to include synchrotron motion, in which case the 6x6 transfer matrix is used instead of the 4x4 matrix. It is then possible to include the contribution of energy spread in the beam to the beam size. For electron storage rings, the equilibrium emittances  $\varepsilon_I$ ,  $\varepsilon_{II}$  (and  $\varepsilon_{III}$  in the longitudinal plane) can be calculated using a method such as that of Chao [4]. For practical purposes, it is often convenient to assume values for  $\varepsilon_I$  and  $\varepsilon_{II}$  based on the design natural emittance of the lattice, and some coupling value.

## EXAMPLE: PEP-II LER

As an illustration of the use of the lattice functions defined in the previous sections, we consider the PEP-II LER lattice. We have calculated the lattice functions using a lattice model based on a particular machine configuration, i.e. including variations in magnet strengths corresponding to a particular operational state.



**Figure 1. Top: vertical beta function around the PEP-II LER interaction region calculated using the normal form analysis and using MAD (the Edwards and Teng method). Bottom: coupling function.**



**Figure 2. Top: vertical beam size around the PEP-II LER IR. Bottom: beam “tilt” (x-y correlation) around the PEP-II LER IR.**

Figure 1 compares the vertical beta function defined in (3), with that calculated in MAD using the same lattice, over a range 80 m either side of the interaction point. MAD calculates coupled beta functions using the method of Edwards and Teng [5] (this method is also the basis of [1]). For the typical case shown, the beta function from MAD ranges from  $-3.5$  km to  $15$  km; the physical interpretation of a negative beta function is unclear.

Figure 1 also shows the coupling function  $\zeta_x$  that gives the contribution of the horizontal emittance to the vertical

beam size. For this lattice, the coupling function has a size comparable to the beta function, suggesting that the horizontal emittance will make a significant contribution to the vertical beam size, particularly since the horizontal emittance is ten times larger than the vertical (25 nm, compared to 2 nm). This is confirmed by comparison with the top plot in Figure 2, which shows the vertical beam size through the same region of the lattice. With the “normal form” definitions, the beam size  $\sigma_y^2$  is simply a linear combination of the beta function and the (square of the) zeta function, with coefficients equal to the emittances. Note that the calculation from (3) using the coupled lattice functions gives excellent agreement with the MAD calculation of the beam size (as it should: the beam size has an unambiguous physical definition).

Finally, the bottom plot in Figure 2 shows a comparison of the correlation  $\langle xy \rangle$ , calculated for the same region of the lattice, from (3) and from MAD. Again, there is excellent agreement between the calculations.

## CONCLUSIONS

We have shown that normal form analysis of the linear single-turn matrix in a storage ring leads to natural definitions of the beta functions for a coupled lattice. It is also possible to define functions (again, in terms of the components of the normalizing matrix) that describe the coupling. Using this approach, we find lattice functions that are always well behaved, by being always positive and finite. The lattice functions defined in this way can be used in simple expressions with the beam emittances to give the second-order moments of the beam distribution. The lattice functions also have simple physical interpretations in terms of the betatron motion of particles in the lattice. For a lattice away from the coupling resonance, there is no ambiguity with associating the lattice functions with the emittances, since they are associated with each other through the betatron tunes. The “normal form” definitions of the lattice functions are now being used in the PEP-II on-line model, to provide convenient descriptions of the lattices under real operating conditions.

## REFERENCES

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