SIMPLE ANALYTIC FORMULAE FOR THE PROPERTIES OF NONSCALING FFAG LATTICES

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Abstract

A hallmark of the "non-scaling" FFAG lattices recently proposed for neutrino factories and muon colliders is that a wide range of momentum is compacted into a narrow radial band; $\alpha = (dL/L)/(dp/p)$ is of order 10^{-3}. This property is associated with the use of F0D0, doublet or FDF triplet lattices in which the F magnet provides a reverse bend. In this paper simple analytic formulae for key lattice properties, such as orbit displacement and path length as a function of momentum, are derived from thin-element models. These confirm the parabolic dependence of path-length on momentum observed with standard orbit codes, reveal the factors which should be adjusted to minimize its variation, and form a useful starting point for the thick-element design. Finally, in the context of a 10-20 GeV/c muon ring, the thin-element formulae are compared against lattice optical properties computed for thick-element systems; the discrepancies are small overall, and most discernible for the triplet lattices.

THIN-ELEMENT MODEL

A simple model for the closed orbits versus momentum in nonscaling FFAG lattices[1] has been proposed and refined by the authors at a recent series of workshops[3, 4, 5]. The model has two components: (1) *Physics* – magnetic elements represented by kicks; and (2) *Geometry* – path lengths from the "law of sines". There have been three stages in the refining of our understanding of this model:

- Under the assumption of equal cell length and quadrupole strength, geometry suggests F0D0 gives the shortest path-length, then doublet, then triplet.
- Under the assumption of equal cell length and equal betatron tunes, a calculation to second order shows that the increased quadrupole strengths of the doublet and triplet (compared to F0D0) exactly compensate the geometric effect so that path-length performance of the three lattice types is identical (at this order).
- Under similar assumptions, but to third order in element strengths, not only are the strengths greater *but also* the strength splitting is greater for the triplet (in particular) and the doublet when compared to the F0D0; and this leads to a reversal of the path-length ranking: triplet, (doublet, F0D0).

However, fixed cell length is a false constraint. The main advantage of the doublet is that the 2nd drift space may be very short, leading to reduced time-of-flight. Note also that magnets in the triplet tend to be longer.

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Elaboration of the Model

We shall employ the subscripts d, f to indicate whether a quantity is associated with a D or F element. Lengths l and quadrupole strengths β are values within the half-cell. Thus the strengths $\beta_d = B_{1d} l_d$ and $\beta_f = B_{1f} l_f$, where B_1 is the field gradient. The element separations l_0, l_1, l_2 are defined in Figs.1,2.

A simple example of the thin-element model is the F0D0. The reference momentum p_c is defined as that whose orbit is on axis in the F-quadrupole; the half-bend angle of the D-sector is θ. At other momenta p the angular kicks ψ are proportional to the strengths β and displacements x:

$$\psi_d = (p_c\theta - \beta_d x_d)/p \ , \ \psi_f = \beta_f x_f/p \ , \ \psi_d + \psi_f = \theta \ . \tag{1}$$

From the "law of sines"

$$\frac{(l_0 + \delta l_0)}{\sin\theta} = \frac{(x_f + l_0\cot\theta)}{\cos\psi_d} = \frac{(x_d + l_0\csc\theta)}{\cos\psi_f} \ . \tag{2}$$

Displacements of closed orbits at D and F elements:

$$x_d \approx \frac{(p-\beta_f l)(p-p_c)\theta}{\beta_d\beta_f l - p\Delta\beta} \ , \ \ x_f \approx \frac{p(p-p_c)\theta}{\beta_d\beta_f l - p\Delta\beta} \ . \tag{3}$$

The cell path length is $\mathcal{L}_0$ at momentum p_c. To lowest order the increment at other momenta p is $\delta\mathcal{L}$:

$$(p-p_c)\frac{\left[\beta_f l[2\beta_d(2p-\beta_f l) - \beta_f(p+p_c)] - 2p^2\Delta\beta\right]}{[\beta_d\beta_f l - p\Delta\beta]^2}\theta^2 \tag{4}$$

Here and throughout $l = l_0, l_2, \lambda$ for F0D0, triplet and doublet respectively, with $\lambda = l_1 l_2/l_0$. Also $\Delta\beta = \beta_d - \beta_f$.

Expressions (3,4) are a global momentum description of behaviour; not merely a local linear indication. If $\Delta\beta = 0$, then the pathlength variation is parabolic in momentum.

Evidently a strength splitting ($\Delta\beta<0$) can be beneficial due to an increase of the size of denominator in (Eq.4), and parabolae are not sacrosanct. However, the strengths β_d, β_f are not free parameters; they are constrained by the tunes and lattice type, as indicated below.

Lattices with D & F Sectors

Thus far we have considered lattices in which the D element is a combined function magnet and the F element is a pure quadrupole (at the reference momentum). We consider now the case that the F element is also a combined function magnet. The two bends are constrained by the closure condition $\theta_d + \theta_f = \theta = \pi/N_c$. Typically the F sector has a reverse bend angle $\theta_f < 0$ and so the D sector must have stronger bending $\theta_d > \theta$.

There is little difference in optical performance between cells with one or two combined-function sector magnets. What is different is the centring, or lack of it, within the F magnet. With a single sector bend (i.e. $\theta_f = 0$), we have no choice in whether the low and high momentum orbits are disposed equally about the reference-momentum orbit. With two sector bends, we gain an additional free parameter, θ_f, and are at liberty to choose how this extra freedom is used. For example, we may set the reference momentum at the minimum of the parabola; but in that case we have lost control of the disposition of the orbits. Alternatively, if p_c "floats", then θ_f can be used to minimise one from several orbit quantities (e.g. centring and aperture).

The analogues of the closed orbit offsets (3) are

$$x_d = (p-p_c)\frac{(p\theta - \beta_f l\theta_d)}{\beta_d\beta_f l - p\Delta\beta}\ ,\ x_f = (p-p_c)\frac{(p\theta + \beta_d l\theta_f)}{\beta_d\beta_f l - p\Delta\beta}\ . \quad (5)$$

Due to space limitations, we give the path-length formula only in the case of equal quadrupole strengths ($\beta_d = \beta_f$):

$$\delta\mathcal{L} = (p-p_c)\theta[(3p-p_c)\theta + 2\beta_f l(\theta_f - \theta_d)]/(\beta_f^2 l)\ . \quad (6)$$

The coefficient of p^2 measures the strength of the dispersion in path lengths, and is equal to $3\theta^2/(\beta_f^2 l)$; which is independent of how the dipole bending is shared between F and D elements. In a higher order calculation where strength splitting becomes important, it is found that concentrating the positive bending in the D sector is beneficial.

Geometry Effect

Assuming equal cell lengths and that all lattice types have equal integrated quadrupole strengths, geometry suggests F0D0 gives the shortest path-length, then doublet, then triplet. This geometrical effect is illustrated in Figs. 1,2. The cell lengths are $\mathcal{L}_0 = 2(l_d + l_f + l_0), 2(l_d + l_f) + (l_1 + l_2), 2(l_d + l_f + l_1 + l_2)$, for F0D0, doublet, triplet, respectively.

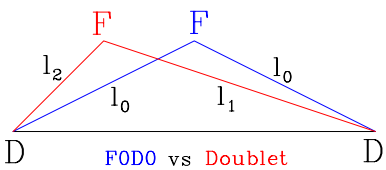

Figure 1: Layout of F0D0 cell with D-sector and F-quad.

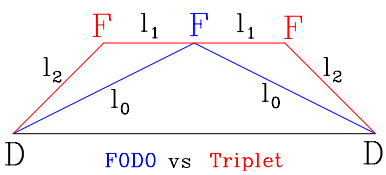

Figure 2: Layout of F0D0 cell with D-sector and F-quad.

ELEMENT STRENGTHS

To maximize the focusing and yet avoid the 1/2-integer resonance, we take the cell phase advance Φ in horizontal and vertical planes to be equal and just under π at the injection momentum. Starting from exact thick-element formulae for the tunes, expansions in powers of $A_{d,f} \equiv \beta_{d,f}/p$ were made to second and higher order as noted below.

F0D0

To 2nd order, $A_d = A_f \equiv A_0$ with

$$A_0^2 = \frac{3(1-\cos\Phi)}{\mathcal{L}_0(l_d + l_f + 3l_0)} \rightarrow \frac{(1-\cos\Phi)}{2l_0}\ . \quad (7)$$

The field gradients are

$$B_{1d} = A_d\, p/(l_d\, c)\ , B_{1f} = B_{1d}(l_d/l_f)\ . \quad (8)$$

To 4th order there is a splitting; in the thin lens limit, $l_d, l_f \rightarrow 0$, one obtains

$$(A_d + A_f) = 2A_0\left[1 + (1-\cos\Phi)(l_d - l_f)^2/(3l_0)^2\right] \quad (9)$$

$$(A_d - A_f) = A_0(1-\cos\Phi)(l_d - l_f)/(3l_0)\ . \quad (10)$$

Triplet

To 2nd order we find the same expression as for the F0D0 case (7) *but* with $3l_0$ substituted by $3l_2$. Evidently, the triplet quadrupole strengths are larger than in the F0D0 case. To 3rd order there is a splitting; in the thin lens limit one obtains:

$$\delta A_f = A_0(1-\cos\Phi)\frac{l_2(2l_1+l_2)}{4(l_1+l_2)^3}\left[l_1 - \frac{2}{3}(l_d - l_f)\right] \quad (11)$$

$$\delta A_d = A_0(1-\cos\Phi)\frac{l_2^2}{4(l_1+l_2)^3}\left[\frac{2}{3}(l_d - l_f) - l_1\right] \quad (12)$$

Doublet

To 2nd order the strength is $A_d = A_f \equiv A_0$ with

$$A_0^2 = \frac{3(1-\cos\Phi)}{[\mathcal{L}_0(l_d + l_f + 3l_0) + 6(l_1 l_2 - l_0^2)]} \rightarrow \frac{(1-\cos\Phi)}{2l_1 l_2} \quad (13)$$

Evidently, the integrated quadrupole strength can be greater than in the F0D0 case because $l_0^2 \geq l_1 l_2$. To 3rd order there is a splitting; in the thin lens limit one obtains:

$$\delta A_{d,f} \rightarrow \pm A_0(1-\cos\Phi)(l_d - l_f)/(6l_0)\ . \quad (14)$$

For $\delta A_d/\delta A_f$ take the positive/negative sign, respectively.

OPTIMUM PATH LENGTH

As noted previously, the different strengths and splittings have a profound effect on lattice performance, as does cell length. We have obtained expressions for the orbit offsets and path lengths with strength splittings as appropriate to the three lattice types; and in each case found conditions to optimize p_c. However, because of space limitations we present here results for the simple case of equal strengths.

Let $\check{\mathcal{L}}, \hat{\mathcal{L}}$ be the minimum, maximum path lengths. The condition of smallest pathlength *variation* is that paths be

equal at the injection and extraction momenta $\check{p}$, $\hat{p}$. The pathlength increment is:

$$\delta\mathcal{L} = (p - p_c)(3p - p_c - 2\beta_f l)\,\theta^2/(\beta_f^2 l)\ . \qquad (15)$$

The condition $\mathcal{L}(\check{p}) = \mathcal{L}(\hat{p})$ is solved for the reference momentum:

$$p_c = [3(\check{p} + \hat{p}) - 2\beta_f l]/4\ . \qquad (16)$$

The minimum path length occurs at momentum

$$p(\check{\mathcal{L}}) = \frac{1}{3}(2p_c + \beta_f l)\ ,\ \ \delta\check{\mathcal{L}} = -\frac{(p_c - \beta_f l)^2}{3\beta_f^2 l}\theta^2\ . \qquad (17)$$

Substituting (16) in (17) we discover that $p(\check{\mathcal{L}}) = (\check{p}+\hat{p})/2$ as is anticipated on the grounds of symmetry. We make the same substitutions in the pathlength to find $\hat{\mathcal{L}} = \mathcal{L}(\check{p}) = \mathcal{L}(\hat{p})$. Hence the total pathlength variation is

$$\hat{\mathcal{L}} - \check{\mathcal{L}} = 3(\hat{p} - \check{p})^2\theta^2/(4\beta_f^2 l)\ . \qquad (18)$$

Under the condition of equal betatron tunes, in the thin lens limit, and to 2nd order in A_d, A_f, we have the relation between F0D0, doublet and triplet quadrupole strengths that

$$\beta_{F0D0}^2 l_0 = \beta_{\rm doublet}^2 \lambda = \beta_{\rm triplet}^2 l_2 = (1 - \cos\Phi)(\check{p})^2/\mathcal{L}_0\ . \qquad (19)$$

Thus, remarkably, the three (optimized) lattices have equivalent pathlength performance when strengths are calculated to 2nd order; at higher order the splittings become important and the picture changes.

Table 1 shows values of the cell length variation $\Delta\mathcal{L} = \hat{\mathcal{L}} - \check{\mathcal{L}}$ from (18), and of x_f at 10 and 20 GeV from (4), for various lattices, compared to those obtained by tracking. The FDF triplet, doublet and F0D01 were presented by Berg[3], the seminal F0D03 by Johnstone[1], and F0D02 by both[2]. Subscripts 10,20 refer to the momentum (GeV/c), while indices F, T denote thin-element and tracked values, respectively. The agreement is very good for F0D0 lattices, less good for doublet and triplet ones, where the close spacing of the magnets makes the thin-lens approach less effective. (Excellent agreement can be recovered by including finite-length magnets in the analytic treatment[5].)

Table 1: Comparison of formulae with tracking

	FDF	doublt	F0D01	F0D02	F0D03
$\mathcal{L}$ (m)	5.18	4.32	5.47	6.00	6.5
N_c	93	101	113	200	314
p_c (GeV/c)	20.1	18.8	18.2	18.9	16.6
$\langle\beta\rangle$ (T)	6.30	5.32	3.61	3.06	1.70
$x_{f10,F}$ (cm)	-7.5	-10.1	-8.2	-6.4	-7.1
$x_{f10,T}$ (cm)	-9.7	-9.9	-8.3	-6.8	-7.2
$x_{f20,F}$ (cm)	0.1	0.6	3.0	1.4	7.5
$x_{f20,T}$ (cm)	-1.9	0.3	3.4	1.8	7.5
$\Delta\mathcal{L}_F$ (mm)	2.51	2.38	1.99	0.85	1.64
$\Delta\mathcal{L}_T$ (mm)	1.84	2.12	2.12	0.90	1.62

Number of Cells N_c

The spread in cell transit time $\delta T = \tau_0\Delta\mathcal{L}/\mathcal{L}_0$ is subject to two requirements, one set by (18) and (19), the other by the minimum energy gain per cell.

$$\frac{\delta T}{\tau_0} = \frac{3}{4}\frac{(\hat{p} - \check{p})^2\,\theta^2}{(\check{p})^2(1 - \cos\Phi)}\ ,\ \ \frac{\delta p}{\omega\,\delta T}\frac{1}{(\hat{p} - \check{p})} \geq W\ . \qquad (20)$$

Here W is a parameter determining the longitudinal acceptance[3], δp is the momentum impulse per cell, and ω is angular radio frequency. So the minimum number of cells N_c is given by:

$$\frac{4}{3}\frac{(1 - \cos\Phi)}{\omega\tau_0}\left[\frac{N_c}{\pi}\right]^2 \geq W\frac{(\hat{p} - \check{p})^3}{(\check{p})^2\delta p}\ . \qquad (21)$$

Typically this formula results in slightly low values for F0D0 and doublet, and slightly high ones for the triplet.

SUMMARY

Based on calculation of the quadrupole strengths to 2nd order we have given expressions for orbit offsets, path lengths, etc., for a variety of lattices and found them to be remarkably similar. Space limitations prohibit the presentation of results to higher order. However, continuing the expansion to third order one concludes:

- For equal tunes $\beta_d \neq \beta_f$.
- The splitting is largest and in the correct direction $(\beta_f > \beta_d)$ for the triplet.
- For equal tunes, the strength splitting is smallest and in the wrong direction $(\beta_d > \beta_f)$ for the F0D0.
- Corollaries: (1) For a triplet, the strong strength-splitting term in the denominator will give a slight tilt to the parabola. (2) For the F0D0, a tune split between horizontal and vertical will increase the strength split and may have a beneficial effect on path length.
- The expansion parameter is largest, and the truncated series approximation least accurate, at the injection momentum. But this is precisely the circumstance under which we specify the tune and design the lattice; and from this derives the deficiencies of formula (21).
- The analogue of (21) for the optimum number of cells when $\beta_d \neq \beta_f$ has been obtained; and is typically an underestimate for the triplet.
- Strength splitting implies modification of the model longitudinal hamiltonian - no longer a pure cubic.

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