COUPLING COEFFICIENTS IN THE INHOMOGENEOUS CAVITY CHAIN

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Abstract

In this paper a mathematical method on the base of a rigorous electrodynamic approach for description of inhomogeneous chain of cylindrical cavities is presented. The form of the obtained for chosen amplitudes set of equations is similar to the set of equations that describe the simple coupled circuit chain. As the cavity have the infinite number of resonant frequencies, to obtain the coupling coefficients one have to solve additional infinite set of linear equations with coefficients that depend on the frequency. By using the developed method for inhomogeneous cavity chain one can take into account the dependence of the coupling coefficients on frequency and geometrical sizes of the cavities in the case of "long-range" coupling.

INTRODUCTION

Inhomogeneous accelerating structures are widely used in the accelerator technology. Application of inhomogeneous structures allows to increase the rate of acceleration and to suppress the transverse instability. In [1] the method of consecutive tuning has been discussed and the comparative analysis of "random" and constant gradient, quasi-constant gradient structures has been done by the next parameters: energy gain, field gradient and damping. Calculations have been done on the base of coupled oscillators model [2] with using of the results from [3]. In [3, 4] the problem of calculation of the coupling coefficients for coupled cylindrical cavities have been solved on the base of a rigorous electrodynamic approach with the help of the method of partial cross-over regions. The homogeneous chain of coupled cylindrical cavities and the structure consisting of two cavities coupled through a center-hole of arbitrary dimensions have been considered. Using of the method of partial cross-over regions for description of inhomogeneous chain of cylindrical cavities results in difficult and clumsy calculations. In this work we propose more simple, from our point of view, method based on rigorous electrodynamic approach for description of inhomogeneous structures.

MATHEMATICAL MODEL OF INHOMOGENEOUS CAVITY CHAIN

Consider the chain of ideally conducting co-axial cylindrical cavities coupled through apertures of the disks with different radii a_i (see Fig. 1). The thickness of each disk is t. The radii and lengths of the cavities we denote by b_i and d_i, correspondingly.

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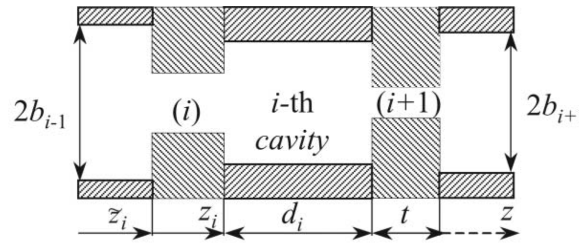

Figure 1: Inhomogeneous chain of coupled cavities.

Electromagnetic field in the cavity we present as superposition of separated modes of a single resonator:

$$E_{z,mp}^{(i)} = \sum_m \sum_p A_{mp}^{(i)} J_0(k_{\perp m}^{b(i)} r) \cos(k_p^{(i)}(z - z_i))$$

$$E_{r,mp}^{(i)} = \sum_m \sum_p A_{mp}^{(i)} \frac{k_p^{(i)}}{k_{\perp m}^{b(i)}} J_1(k_{\perp m}^{b(i)} r) \sin(k_p^{(i)}(z - z_i)) \tag{1}$$

$$H_{\varphi,mp}^{(i)} = \sum_m \sum_p B_{mp}^{(i)} \frac{-i\varepsilon_0 \omega_{mp}^{(i)}}{k_{\perp m}^{b(i)}} J_1(k_{\perp m}^{b(i)} r) \cos(k_p^{(i)}(z - z_i))$$

where $A_{mp}^{(i)}$ is the amplitude and $\omega_{mp}^{(i)}$ is the resonant frequency of axially-symmetric E_{0mp}-mode in the i–th cavity; $B_{mp}^{(i)} = A_{mp}^{(i)} \omega / \omega_{mp}^{(i)}$.

Electromagnetic field in the aperture we present as superposition of eigen oscillations of uniform waveguide:

$$E_{z,s}^{(i)} = \sum_s J_0\left(k_{\perp s}^{a(i)} r\right) \left[C_{1s}^{(i)} \exp(-h_s^{(i)}(z - \mathfrak{z}_i)) + \right.$$
$$\left. + C_{2s}^{(i)} \exp(h_s^{(i)}(z - \mathfrak{z}_i))\right]$$
$$E_{r,s}^{(i)} = \sum_s J_1\left(k_{\perp s}^{a(i)} r\right) \left[C_{1s}^{(i)} \exp(-h_s^{(i)}(z - \mathfrak{z}_i)) - \right.$$
$$\left. - C_{2s}^{(i)} \exp(h_s^{(i)}(z - \mathfrak{z}_i))\right] \frac{h_s^{(i)}}{k_{\perp s}^{a(i)}} \tag{2}$$
$$H_{\varphi,s}^{(i)} = \sum_s J_1\left(k_{\perp s}^{a(i)} r\right) \left[C_{1s}^{(i)} \exp(-h_s^{(i)}(z - \mathfrak{z}_i)) + \right.$$
$$\left. + C_{2s}^{(i)} \exp(h_s^{(i)}(z - \mathfrak{z}_i))\right] \frac{-i\varepsilon_0 \omega}{k_{\perp s}^{a(i)}}$$

Tangential components of electric field at the left and right boundaries of the i–th disk aperture are expanded into the series with the complete set of the first order Bessel functions:

$$E_{r+}^{(i)} = \sum_s w_{s+}^{(i)} J_1\left(k_{\perp s}^{a(i)} r\right)$$
$$E_{r-}^{(i)} = \sum_s w_{s-}^{(i)} J_1\left(k_{\perp s}^{a(i)} r\right) \tag{3}$$

where $0 < r < a_i$. Sign (+) refers to the right boundary of the disk and sign (-) – to the left one, correspondingly. As the tangential component of electric field is continuous at the both boundaries of the i–th disk, coefficients C and w are connected via the following equations:

$$C_{1s}^{(i)} = \frac{k_{\perp s}^{a(i)}}{h_s^{(i)}} \frac{w_{s-}^{(i)} \exp(h_s^{(i)} t) - w_{s+}^{(i)}}{2sh(h_s^{(i)} t)}$$
$$C_{2s}^{(i)} = \frac{k_{\perp s}^{a(i)}}{h_s^{(i)}} \frac{w_{s-}^{(i)} \exp(-h_s^{(i)} t) - w_{s+}^{(i)}}{2sh(h_s^{(i)} t)} \qquad (4)$$

Coefficients in expansion (1) are determined by the tangential components of electric field $E_{r+}^{(i)}$ and $E_{r-}^{(i+1)}$:

$$A_{mp}^{(i)}\left(\omega^2 - \omega_{mp}^{(i)2}\right) = \frac{2\pi\omega_{mp}^{(i)}}{iN_{mp}^{(i)}} \times$$
$$\left[\int_0^{a_{i+1}} E_{r-}^{(i+1)} H_{\varphi}^{(i)*}(r, \tilde{z}_{i+1}) r dr - \int_0^{a_i} E_{r+}^{(i)} H_{\varphi}^{(i)*}(r, z_i) r dr\right] \qquad (5)$$

where $N_{mp}^{(i)}$ is the norm of $E_{0\mathrm{mp}}$-mode in the i–th cavity.

As tangential component of magnetic field is continuous at the both boundaries of the i–th disk aperture ($0 < r < a_i$), one can write the equations for the coefficients $A_{mp}^{(i-1)}$, $A_{mp}^{(i)}$, $w_{s-}^{(i)}$, $w_{s+}^{(i)}$. The left and the right parts of these equations are functions of variable r. Since the functions are equal the expansion coefficients of these functions with the complete set of orthogonal functions $J_1\left(k_{\perp n}^{a(i)} r\right)$ are equal too:

$$\sum_m \sum_p \frac{(-1)^p \theta_m^{(i-1,+)} J_0\left(\theta_m^{(i-1,+)}\right)}{k_{\perp m}^{b(i-1)}\left(\lambda_n^2 - \theta_m^{(i-1,+)2}\right)} A_{mp}^{(i-1)} =$$
$$= \frac{J_1(\lambda_n)}{2h_n^{(i)}} \frac{w_{n-}^{(i)} ch\left(h_n^{(i)} t\right) - w_{n+}^{(i)}}{sh\left(h_n^{(i)} t\right)} \qquad (6.1)$$

$$\sum_m \sum_p \frac{\theta_m^{(i,-)} J_0\left(\theta_m^{(i,-)}\right)}{k_{\perp m}^{b(i)}\left(\lambda_n^2 - \theta_m^{(i,-)2}\right)} A_{mp}^{(i)} =$$
$$= \frac{J_1(\lambda_n)}{2h_n^{(i)}} \frac{w_{n-}^{(i)} - w_{n+}^{(i)} ch\left(h_n^{(i)} t\right)}{sh\left(h_n^{(i)} t\right)} \qquad (6.2)$$

where $\theta_m^{(i-1,+)} = k_{\perp m}^{b(i-1)} a_i$, $\theta_m^{(i,-)} = k_{\perp m}^{b(i)} a_i$, $n = 1, 2, \ldots$.

From Eqs. 4, 5 we express all the coefficients A_{mp}, except one, for example, A_{10}, via $w_{s\pm}$. Then, from Eqs. 6 we obtain inhomogeneous set of algebraic equations for $w_{s\pm}$:

$$\tilde{w}_{n+}^{(i)} + 2\sum_s \left(f_n^{(i)} \left[T_{ns}^{(i,-)} \tilde{w}_{s+}^{(i)} - \tilde{T}_{ns}^{(i,+)} \tilde{w}_{s-}^{(i+1)}\right] + \right.$$
$$\left. + \tilde{f}_n^{(i)} \left[T_{ns}^{(i-1,+)} \tilde{w}_{s-}^{(i)} - \tilde{T}_{ns}^{(i-1,-)} \tilde{w}_{s+}^{(i-1)}\right] \right) = \qquad (7.1)$$
$$= 6\pi\left(\tilde{f}_n^{(i)} \frac{A_{10}^{(i-1)} J_0\left(\theta_1^{i-1,+}\right)}{\lambda_n^2 - \theta_1^{(i-1,+)2}} - f_n^{(i)} \frac{A_{10}^{(i)} J_0\left(\theta_1^{i,-}\right)}{\lambda_n^2 - \theta_1^{(i,-)2}} \right)$$

$$\tilde{w}_{n-}^{(i)} + 2\sum_s \left(f_n^{(i)} \left[T_{ns}^{(i-1,+)} \tilde{w}_{s-}^{(i)} - \tilde{T}_{ns}^{(i-1,-)} \tilde{w}_{s+}^{(i-1)}\right] + \right.$$
$$\left. + \tilde{f}_n^{(i)} \left[T_{ns}^{(i,-)} \tilde{w}_{s+}^{(i)} - \tilde{T}_{ns}^{(i,+)} \tilde{w}_{s-}^{(i+1)}\right] \right) = \qquad (7.2)$$
$$= 6\pi\left(f_n^{(i)} \frac{A_{10}^{(i-1)} J_0\left(\theta_1^{i-1,+}\right)}{\lambda_n^2 - \theta_1^{(i-1,+)2}} - \tilde{f}_n^{(i)} \frac{A_{10}^{(i)} J_0\left(\theta_1^{i,-}\right)}{\lambda_n^2 - \theta_1^{(i,-)2}} \right)$$

where $T_{ns}^{(i,\pm)} = \frac{\pi a_{i+1,i}}{b_i} \sum_m \frac{\theta_m^{(i,\pm)3} J_0^2\left(\theta_m^{(i,\pm)}\right) E_m^{(i,\pm)} \chi_m^{-1}}{\left(\lambda_n^2 - \theta_m^{(i,\pm)2}\right)\left(\lambda_s^2 - \theta_m^{(i,\pm)2}\right)}$,

$$\tilde{T}_{ns}^{(i,\pm)} = \frac{\pi a_{i+1,i}}{b_i} \sum_m \frac{\theta_m^{(i,\pm)2} \theta_m^{(i,\mp)} J_0\left(\theta_m^{(i,\pm)}\right) J_0\left(\theta_m^{(i,\mp)}\right) \tilde{E}_m^{(i,\mp)}}{\chi_m \left(\lambda_n^2 - \theta_m^{(i,\mp)2}\right)\left(\lambda_s^2 - \theta_m^{(i,\pm)2}\right)}$$

$f_n^{(i)} = h_n^{(i)} a_i ch(h_n^{(i)} t) / sh(h_n^{(i)} t)$, $\tilde{f}_n^{(i)} = h_n^{(i)} a_i / sh(h_n^{(i)} t)$

$$E_m^{(i,\pm)} = \begin{cases} \dfrac{cth\left(d_i \sqrt{\theta_m^{(i,\pm)2} - \Omega_{i+1,i}^2} / a_{i+1,i}\right)}{\sqrt{\theta_m^{(i,\pm)2} - \Omega_{i+1,i}^2}}, & m \neq 1 \\ \dfrac{cth\left(d_i \sqrt{\theta_m^{(i,\pm)2} - \Omega_{i+1,i}^2} / a_{i+1,i}\right)}{\sqrt{\theta_m^{(i,\pm)2} - \Omega_{i+1,i}^2}} - \Delta, & m = 1 \end{cases}$$

$$\tilde{E}_m^{(i,\pm)} = \begin{cases} \dfrac{sh^{-1}\left(d_i \sqrt{\theta_m^{(i,\pm)2} - \Omega_{i+1,i}^2} / a_{i+1,i}\right)}{\sqrt{\theta_m^{(i,\pm)2} - \Omega_{i+1,i}^2}}, & m \neq 1 \\ \dfrac{sh^{-1}\left(d_i \sqrt{\theta_m^{(i,\pm)2} - \Omega_{i+1,i}^2} / a_{i+1,i}\right)}{\sqrt{\theta_m^{(i,\pm)2} - \Omega_{i+1,i}^2}} - \Delta, & m = 1 \end{cases}$$

$\Delta = a_{i+1,i} / d_i \left(\theta_1^{(i,\pm)2} - \Omega_{i+1,i}^2\right)$, $\Omega_{i+1,i} = a_{i+1,i}\omega / c$,

$\chi_m = J_1^2(\lambda_m)\lambda_m \pi / 2$, $\tilde{w}_{s\pm}^{(i)} = 3\pi J_1(\lambda_s) w_{s\pm}^{(i)}$.

Since $w_{s\pm}$ are the expansion coefficients of tangential at the disk aperture cross-section component of the electric field, Eqs. 7 are the interaction equations for the fields defined in the circular regions. There are several interesting results which may be obtained from Eqs. 7. One is that the fields of only four circular regions interact directly: at the left and right boundaries of the i–th disk aperture – $\tilde{w}_{s-}^{(i)}$, $\tilde{w}_{s+}^{(i)}$, at the right boundary of the i–1–th disk aperture – $\tilde{w}_{s+}^{(i-1)}$ and at the left boundary of the i+1–th disk aperture – $\tilde{w}_{s-}^{(i+1)}$. It follows from the fact that i–th cavity contacts directly only with two neighboring cavities: i–1–th and i+1–th. Another important result obtainable from Eqs. 7 is that the interaction of fields in adjacent apertures is described by the terms, which contain factors $\tilde{T}_{ns}^{(i,+)}$, $\tilde{T}_{ns}^{(i-1,-)}$. It can be shown that $\tilde{T}_{ns}^{(i,+)}$, $\tilde{T}_{ns}^{(i-1,-)} \to 0$ when $a_i \to 0$ and $t = 0$. At the same time factors in terms, which describe fields interaction at the left and right boundaries of single aperture – $T_{ns}^{(i,-)}$, $T_{ns}^{(i-1,+)}$, tend to constant values independent of a_i when $a_i \to 0$ and $t = 0$.

The solution to the set of linear algebraic equations (7) can be written in the following form:

$$\tilde{w}_{s\pm}^{(j)} = \sum_{n=-N}^{N} \zeta_{s\pm}^{(j,i+n)} A_{10}^{(i+n)} \tag{8}$$

The choice of the value of N depends on the fact how many couplings of the i–th cavity with another ones we want to take into account. Index j takes the values $i–N+1$, $i–N+2$, ... , $i–1$, i, $i+1$, ... , $i+ N$, correspondingly. For paired couplings (each cavity is coupled only with adjacent ones) $N = 1$. Substituting expression (8) in Eqs. 7 one can obtain inhomogeneous set of linear algebraic equations for $\zeta_{s\pm}^{(j,i+n)}$:

$$\zeta_{n+}^{(j,i+n)} + 2\sum_s \Big(f_n^{(i)} \Big[T_{ns}^{(i,-)} \zeta_{s+}^{(j,i+n)} - \tilde{T}_{ns}^{(i,+)} \zeta_{s-}^{(j+1,i+n)} \Big] + \\ + \tilde{f}_n^{(i)} \Big[T_{ns}^{(i-1,+)} \zeta_{s-}^{(j,i+n)} - \tilde{T}_{ns}^{(i-1,-)} \zeta_{s+}^{(j-1,i+n)} \Big] \Big) = \\ = 6\pi \left(\tilde{f}_n^{(i)} \frac{\delta_{i-1,i+n} J_0\big(\theta_1^{i-1,+}\big)}{\lambda_n^2 - \theta_1^{(i-1,+)2}} - f_n^{(i)} \frac{\delta_{i,i+n} J_0\big(\theta_1^{i,-}\big)}{\lambda_n^2 - \theta_1^{(i,-)2}} \right) \tag{9.1}$$

$$\zeta_{n-}^{(j,i+n)} + 2\sum_s \Big(f_n^{(i)} \Big[T_{ns}^{(i-1,+)} \zeta_{s-}^{(j,i+n)} - \tilde{T}_{ns}^{(i-1,-)} \zeta_{s+}^{(j-1,i+n)} \Big] + \\ + \tilde{f}_n^{(i)} \Big[T_{ns}^{(i,-)} \zeta_{s+}^{(j,i+n)} - \tilde{T}_{ns}^{(i,+)} \zeta_{s-}^{(j+1,i+n)} \Big] \Big) = \\ = 6\pi \left(f_n^{(i)} \frac{\delta_{i-1,i+n} J_0\big(\theta_1^{i-1,+}\big)}{\lambda_n^2 - \theta_1^{(i-1,+)2}} - \tilde{f}_n^{(i)} \frac{\delta_{i,i+n} J_0\big(\theta_1^{i,-}\big)}{\lambda_n^2 - \theta_1^{(i,-)2}} \right) \tag{9.2}$$

where $n = -N \dots 0 \dots N$; $j = i–N+1, i–N+2, \dots , i–1, i, i+1, \dots , i+ N$.

From Eq. 8 one can deduce that the electric field tangential component in the circular regions, through which i–th cavity is connected with other elements of the structure under consideration, are only determined via the E_{010}-mode amplitudes in the cavities. Eq. 5 for E_{010}-mode amplitude in the i–th cavity will have the form:

$$A_{10}^{(i)}\Big(\omega^2 - \omega_{10}^{(i)2}\Big) = \omega_{10}^{(i)2} \sum_{n=-N}^{N} A_{10}^{(i+n)} \times \\ \Big(\varepsilon_+^{(i)} \Lambda_{i+}^{(i+1,i+n)} - \varepsilon_-^{(i)} \Lambda_{i-}^{(i,i+n)}\Big) \tag{10}$$

where $\varepsilon_\pm^{(i)}$ are the well known coupling coefficients derived on the basis of quasi-static approximation (see, for example, [2])which are given by

$$\varepsilon_\pm^{(i)} = \frac{2}{3\pi J_1^2(\lambda_1)} \frac{a_{i+1,i}^3}{b_i^2 d_i} \tag{11}$$

The coupling coefficients Λ have frequency dependence. They are given by

$$\Lambda_{i+}^{(i+1,i+n)} = J_0\Big(\theta_1^{(i,+)}\Big) \sum_s \frac{\zeta_{s-}^{(i+1,i+n)}}{\lambda_s^2 - \theta_1^{(i,+)2}} \tag{12.1}$$

$$\Lambda_{i-}^{(i,i+n)} = J_0\Big(\theta_1^{(i,-)}\Big) \sum_s \frac{\zeta_{s+}^{(i,i+n)}}{\lambda_s^2 - \theta_1^{(i,-)2}} \tag{12.2}$$

It is necessary to choice the number of terms in sum on n in Eq. 10 equal the number of equations (9). Thus, the problem of coupled cavities has been rigorously reduced to the problem of the coupling of electric fields, which are determined in circular regions.

Eqs. 10 are similar to the equations of coupled oscillators. Influence of non-resonant fields and “long-range” couplings on the characteristics of the structure is taking into account by calculating the coupling coefficients Λ.

CONCLUSION

Thus, mathematical model for description of inhomogeneous chain of cylindrical cavities is developed. Coupling coefficients in the inhomogeneous cavity chain can be calculated with definite accuracy for the structure with arbitrary parameters by taking into account the “long-range” couplings.

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