# MATLAB BASED TPSA TOOLBOX FOR THE PARTICLE MAPPING THROUGH THREE-DIMENSIONAL MAGNETIC FIELDS

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## Abstract

Based on the MATLAB's object-oriented programming, a truncated power series algebra (TPSA) toolbox has been developed. The TPSA toolbox as a differential algebra has been applied to realize the algorithm of particle mapping through three-dimensional magnetic field configurations. The capability of symbolic calculation by using this MATLAB-based TPSA toolbox can be used for the theoretical simulation and modelling in accelerator physics. Employing the TPSA toolbox in MATLAB environment, one can model a real accelerator or storage ring. In this paper, the method of symplectic mapping of three-dimensional magnetic fields is introduced and the structure of TPSA toolbox is presented. Applications of TPSA toolbox in the symplectic mapping of three-dimensional magnetic fields are demonstrated as well.

## INTRODUCTION

Transverse magnetic field in accelerator applications can be conveniently represented with a longitudinal component of vector potential. It makes the treatment of the particle mapping through such two dimensional magnetic fields easily. Solenoids and wigglers were also widely used in the high-energy colliders to modify the damping partition numbers for attaining high quality electron beams. Strong and complex field configurations, super-conducting magnets and elliptic polarization insertion devices, are recently developed for the third or future generation light sources to generate hard X-ray and polarized lights in high performance storage rings. In these devices, the three-dimensional magnetic field can no longer be represented by a single component of the vector potential. A vector potential with a minimum of two components under a properly chosen gauge can be used to describe such a system. However, the corresponding Hamiltonian for particle motion becomes more difficult to solve. Fortunately, a method for extracting the symplectic mapping in systems with three-dimensional magnetic fields [1] had been developed to solve such a difficult task.

The realization of the above algorithm needs the differential algebras. Refer to the Zlib [2], a MATLAB based TPSA toolbox has been developed to carry out this method. It will be described in this paper.

## MATLAB BASED TPSA TOOLBOX

Object-oriented programming (OOP) has become one of the important characters of advanced programming languages. Like almost recent programming languages, MATLAB 6 [3] started providing the OOP by a relatively simple way. One can create a directory named "@class" to declare and to define the constructor and all overloading functions of this "class" under this specified directory.

MATLAB based TPSA toolbox, which is a directory named "TPSA", includes two parts (sub-directories), "OSIP" and "@TPS", and both of them are all developed with the M files of MATLAB. The one-step index pointer (OSIP) is a global object and is taken as the nerves of TPSA for quickly accessing the coefficients of truncated power series (TPS). A key-function, "*OSIP_Nerve(nv, mo, cd)*", is provided to build up the OSIP. The other part of the TPSA toolbox is the TPS class. In which, all the major functions of TPSA have been developed including the constructor, display, overloading operators, and many Lie algebra related functions.

Following is a simplified example to obtain the map of a sector magnet. It builds up an OSIP with 5-variable, maximum order 4, and 2 canonical dimensions. Once the OSIP has been built up, one can start to use the TPS methods (the operators or functions in MATLAB's terminology). It is a symbolic-like programming.

```
% Example of TPSA (Sector)
global OSIP
MaxOrder = 4;
CanDim = 2;
NumOfParameter = 1;
NumOfVar = 2*CanDim +NumOfParameter;
OSIP_Nerve(NumOfVar, MaxOrder, CanDim);
X = VariableOfTPS(TPS, 1);
PX = VariableOfTPS(TPS, 2);
Y = VariableOfTPS(TPS, 3);
PY = VariableOfTPS(TPS, 4);
Delta = VariableOfTPS(TPS, 5);
As =-B0*(X+X^2/r/2);
H = -As/Br-(1+X/r)*sqrt((1+Delta)^2-PX^2-PY^2);
hs = -L*H;
M = TPS2DepritLie(hs);
```

## APPLICATIONS OF TPSA TOOLBOX

Using the TPSA toolbox, we can calculate the maps of magnet elements used in accelerators. These magnets are treated and classified as transverse field model, solenoid, and insertion devices in the following. The dimensionless Hamiltonian in Frenet-Serret coordinate system is

$$H = -a_s - \left(1 + \frac{x}{\rho}\right)\sqrt{(1+\delta)^2 - (p_x - a_x)^2 - (p_y - a_y)^2}$$

$$a_{x,y,s} \equiv \frac{qA_{x,y,s}}{P_{m0}} = \frac{A_{x,y,s}}{B_0\rho}, p_{x,y} \equiv \frac{P_{x,y}}{P_{m0}}, \delta \equiv \frac{(P_m - P_{m0})}{P_{m0}}$$

where $P_{x,y}$ are canonical conjugate momenta, $P_m$ is the magnitude of the mechanical momentum and $P_{m0}$ is the

reference mechanical momentum, $q$ is the charge of particle, $A_{x,y}$ are components of the vector potential and $\rho$ is the radius of curvature.

## Transverse Field Model

The magnetic fields, associated vector potentials, and Hamiltonian of the quadrupole, sector, and combined function sector magnets are

● Quadrupole

$$\begin{cases} B_x = B_1 y, B_y = B_1 x \\ A_s = B_1\left(y^2 - x^2\right)/2 \\ H = -a_s - \sqrt{(1+\delta)^2 - p_x^2 - p_y^2} \end{cases}$$

● Sector

$$\begin{cases} B_y = B_0 \\ A_s = -B_0\left(x + x^2/2\rho\right) \\ H = -a_s - (1+x/\rho)\sqrt{(1+\delta)^2 - p_x^2 - p_y^2} \end{cases}$$

● Combined Function Sector

$$\begin{cases} B_y = B_0 + B_1 x, B_x = B_1 y \\ A_s = -B_0\left(x + x^2/2\rho\right) + B_1(1+x/\rho)\left(y^2 - x^2\right)/2 + B_1 x^3/6\rho \\ H = -a_s - (1+x/\rho)\sqrt{(1+\delta)^2 - p_x^2 - p_y^2} \end{cases}$$

Note that there is no first order term in the above dimensionless Hamiltonians and the meaningless constant term of Hamiltonian can be ignored such that the order of Hamiltonian represented by TPS is equal or higher than two. The transfer maps of the Hamiltonian system can be presented in the Deprit-Lie type

$$m = e^{-:L\cdot H:} = e^h \cdot e^f$$

where $exp(h)$ is the linear part, which can be represented as a transfer matrix, and $exp(f)$ is the nonlinear part.

## Solenoid

The magnetic field, vector potential, and dimensionless Hamiltonian of a solenoid are

$$\begin{cases} B_s = b_0 \\ A_x = -\dfrac{b_0 y}{2}, A_y = \dfrac{b_0 x}{2} \\ H = -\sqrt{(1+\delta)^2 - (p_x - a_x)^2 - (p_y - a_y)^2} \end{cases}$$

For the Hamiltonian, there is no first order term and the constant term is ignored. The linear part of the solenoid map, calculated by TPSA method, matches the matrix

$$\begin{pmatrix} \cos^2\theta & \sin\theta\cos\theta/g & \sin\theta\cos\theta & \sin^2\theta/g \\ -g\sin\theta\cos\theta & \cos^2\theta & -g\sin^2\theta & \sin\theta\cos\theta \\ -\sin\theta\cos\theta & -\sin^2\theta/g & \cos^2\theta & \sin\theta\cos\theta/g \\ g\sin^2\theta & -\sin\theta\cos\theta & -g\sin\theta\cos\theta & \cos^2\theta \end{pmatrix}$$

where $g = b_0/2B_0\rho$, $\theta = gL$, and $L$ is the solenoid length.

## Insertion Devices

Consider an ideal insertion device with harmonic $n=1$, the magnetic field is as following

$$B_x = -B\frac{k_x}{k_y}\sin(k_x x)\sinh(k_y y)\cos(k_s s)$$

$$B_y = B\cos(k_x x)\cosh(k_y y)\cos(k_s s)$$

$$B_s = -B\frac{k_s}{k_y}\cos(k_x x)\sinh(k_y y)\sin(k_s s)$$

The Maxwell equations require that $k_x^2 + k_s^2 = k_y^2$. For a typical insertion device with pole-width large enough, one may take $k_x \approx 0$. Otherwise, the roll-off effect may cause the horizontal tune shift. Using the TPSA toolbox in the modelling of insertion devices, we can obtain the roll-off effect, i.e. horizontal tune shift extracted from the transfer matrix (linear part of ID's TPS map).

Let $n_e$ be the number of end-pole at each side of a finite insertion device and $n_f$ be the number of full poles, i.e., the total number of poles is $2n_e + n_f$. The magnetic field strengths of end-poles are always adjusted so that the first- and second- integrations of magnetic field are both zero to match the design goal, no net angular and orbital displacements.

Modelling such an insertion device, we formularize the vector potential of the magnetic field with two transverse components. We require the vector potential of all poles keep the continuity along the trajectory of reference particle at all interfaces among full-poles, end-poles, free-space by adjusting the constant term. Because the vector potential is a function of the longitudinal coordinate $s$, we cut each pole of the insertion device into N slices. Then one may track the trajectory of reference particle from the entry of each slice to its mid-point and from the mid-point to the end of this slice by solving the canonical equations of motion, i.e., the Lorentz force equation. At the mid-point $\bar{s}_k$ of each slice $k$, the Hamiltonian is expanded as a TPS around the reference particle such that there is no first order term in the Hamiltonian. Take the mid-point Hamiltonian of each slice as the averaged Hamiltonian of this slice. The map of the $k$-th slice is presented as

$$m_k = e^{-:\Delta s\cdot H(\bar{s}_k):} = e^{h_k/2}\cdot e^{f_k}\cdot e^{h_k/2}$$

Simultaneously, the map of each slice is integrated with the previous integrated map. Then the TPS map of the insertion device is obtained by integrating the maps through all the $N_{ID} = N(2n_e + n_f)$ slices.

$$m = \prod_{k=1}^{N_{ID}} m_k = \prod_{k=1}^{N_{ID}} \left(e^{h_k/2}\cdot e^{f_k}\cdot e^{h_k/2}\right) = e^h \cdot e^f .$$

For a practice, an ideal super-conducting wavelength shifter (SWLS) with three poles, of which pole-width is 0.2 m ($k_s = 15.7080$) is demonstrated. The peak-field ratio of these three poles is $0.5:-1:0.5$ and the peak-field of the central pole is $B = -5.9446$ Tesla. Reference particle, a 1.5 GeV electron, i.e., the momentum rigidity $|B_0\rho| = 5.003461$, is used to pass through such an insertion devices. We can calculate the map of this SWLS and study the beam dynamic effects. Figure1 shows the

magnetic field and vector potential of SWLS and the trajectory of reference particle.
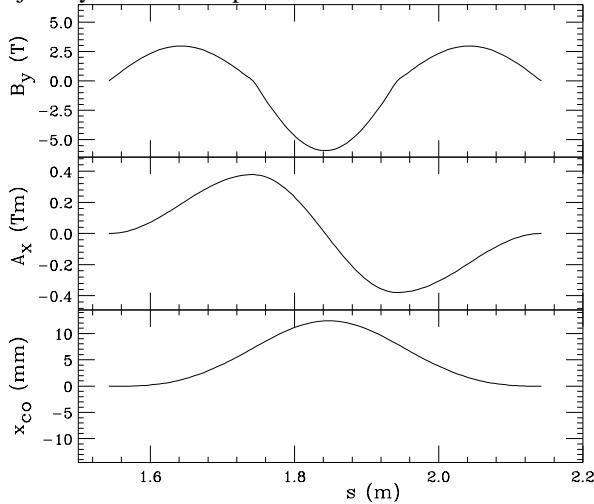


Figure 1: It shows the magnetic field and vector potential of the ideal SWLS only with first harmonic $n=1$ and the trajectory of reference particle passing through this SWLS.

For the $k_x = 0$ case, the field is uniform in horizontal direction. The effective focusing strength calculated by the estimated formula is

$$K_{eff} = 0.5 \cdot (B / B_0\rho)^2 \cdot (1 + 2 \cdot 0.25) / 3 = 0.353$$

The transfer matrix $R$ (5-by-5) obtained from the linear part of the SWLS TPS map is

$$R = \begin{pmatrix} 1 & 0.602157 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0.936771 & 0.583368 & 0 \\ 0 & 0 & -0.209920 & 0.936771 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

And the effective vertical focusing strength 0.360 is extracted from the transfer matrix.

For the case that the magnetic field is roll-off in the horizontal direction, the fitted $k_x = 3.6$ is used in our calculation. From the obtained transfer matrix, the roll-off effect can be observed. Following transfer matrix $R$ is the result that the Hamiltonian was expanded along the longitudinal axis and its constant and first order terms were ignored.

$$R = \begin{pmatrix} 1.003535 & 0.602157 & 0 & 0 & -0.000030 \\ 0.011741 & 1.003535 & 0 & 0 & -0.000101 \\ 0 & 0 & 0.930003 & 0.581545 & 0 \\ 0 & 0 & -0.232304 & 0.930003 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

When we expanded the Hamiltonian along the trajectory of reference particle passing through the SWLS, one can see that the feed-down effect caused by the orbit offset perturbs the transfer matrix $R$ as the following:

$$R = \begin{pmatrix} 1.003645 & 0.603474 & 0 & 0 & -0.000096 \\ 0.011710 & 1.003409 & 0 & 0 & -0.000202 \\ 0 & 0 & 0.929950 & 0.581337 & 0 \\ 0 & 0 & -0.232183 & 0.930184 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

The result shows that the feed-down effect is not serious in the SWLS case even though the trajectory of reference particle has larger horizontal offset than normal wiggler. Effective focusing strength and tune-shift were estimated by the following formula.

$$K = -R(i+1,i) / R(i,i+1), i = 1 \text{ or } 3$$

$$\Delta\nu = \overline{\beta} \cdot K \cdot L / 4\pi$$

With the averaged beta functions at the SWLS,

$$\overline{\beta_x} = 11.17, \quad \overline{\beta_y} = 3.33$$

we have the effective focusing strengths and tune-shifts

$$K_x = -0.0194, \quad K_y = 0.400$$

$$\Delta\nu_x = -0.0103, \quad \Delta\nu_y = 0.0635$$

Comparing with the measured tune-shifts

$$\Delta\nu_x = -0.01392, \quad \Delta\nu_y = 0.0504$$

of the SWLS in NSRRC TLS storage ring, the modelling only with the first harmonic $n=1$ approximation can approach the real SWLS about 74%.

## Particle Tracking and Map Analysis

Once the one-turn-map of a storage ring has been achieved by integrating the maps element-by-element. One can apply the map to do the particle tracking by using either the integrable polynomial or nPB methods and finally representing the tracking result in TPS. One can also analyze the one-turn-map by transfer it to some suitable space (action-angle variable space) with suitable basis (resonance basis).

## CONCLUSION AND DISCUSSION

TPSA is a technique to approach the analytic solution of differential algebra. We are developing a MATLAB-based TPSA toolbox, which can be used to obtain the maps of general three-dimensional magnetic fields used in accelerators.

Associated with the commercial software MATLAB, the development of TPSA toolbox provides accelerator physicists a way to analyze the properties of particle beam-lines.

## ACKNOWLEDGEMENT

## REFERENCES

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