

# NEW DISCRETIZATION SCHEME FOR WAKE FIELD COMPUTATION IN CYLINDRICALLY SYMMETRIC STRUCTURES

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## Abstract

Collective effects due to wake fields are a limiting factor in almost every new front line accelerator. Since the early 80's computer codes such as TBCI [1] and MAFIA [2] have been developed for computing wake fields in realistic accelerator structures. With the advent of linear collider studies and small wavelength FEL projects these codes had to face a severe limitation. For the very short bunches in these new accelerators combined with the need for an analysis of very long sections the discrete dispersion became a serious drawback. This effect of having only discrete field values rather than continuous ones can be overcome by special algorithms such as semi-implicit integrators as used e.g. in the wake field code ECHO [3]. In this paper we present a new explicit approach which combines the advantage of explicit algorithms (fast) with the absence of dispersion in beam direction.

## INTRODUCTION

New FEL projects demand very short bunches for their different physical applications. This results in the need of analysing very long structures wherein discrete dispersion is a serious drawback of available numerical codes.

The Courant criterion limits the time step of explicit numerical algorithms in time domain with respect to the spatial discretisation of the computational domain. Because of that the time step in FDTD algorithms is smaller than the mesh step and for short bunches the calculated wake potential shows discrete dispersion error effects.

In order to overcome this drawback in calculations for short bunches, the properties of the mesh should be used in an adequate way. It is well known that for maximal time step allowed by the Courant limit, the FDTD algorithm has zero dispersion along the grid diagonals. Hence by rotating the mesh the dispersion in the direction of the bunch's motion can be avoided and a powerful code to calculate wake potentials for very short bunches in long structures can be developed.

In this paper we consider a cylindrical symmetric problem with an on-axis bunch. It can be reduced to a 2D problem in the  $r$ - $z$ -plane. By rotating the mesh by an angle of  $45^\circ$  the diagonal of the mesh cells can be aligned to the structure's axis. At each mesh point a local coordinate system can be located. This is referred to as  $u$ - $v$ -system (figure 1). The spacing  $\Delta h$  of the mesh is equidistant and equal in the  $u$  and  $v$  direction. With this spacing  $\Delta h$  the time step for our explicit algorithm will be limited according to the relation

$$\Delta t \leq \frac{1}{\sqrt{2}} \cdot \Delta h$$

A moving mesh can be implemented easily. The scheme will produce very low dispersion effect errors due to the fact that the method has zero dispersion in the direction of the beam propagation. An additional advantage is the fact that for the moving mesh no interpolation will be needed.

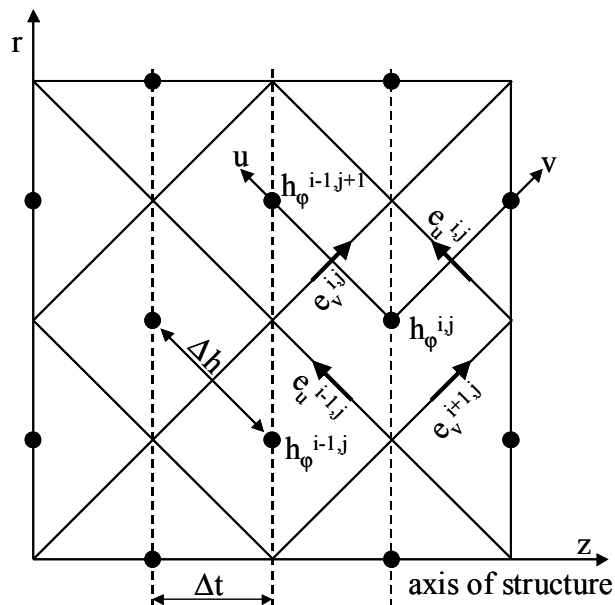


Figure 1: Time step and mesh step

The bunch itself is inserted by the scattered field formulation. The exciting fields of the bunch are introduced at the boundary of the computational domain only. This is advantageous compared to a direct method of inserting the current into the equations. A direct discretisation of the current would result in a zigzag pattern which is not very favourable.

## ALGORITHM

We start from the well known boundary value problem in cylindrical coordinates

$$\frac{\partial h_\phi}{\partial t} = -\frac{1}{\mu_\phi} \left( \frac{\partial e_r}{\partial z} - \frac{\partial e_z}{\partial r} \right)$$

$$\frac{\partial e_r}{\partial t} = \frac{1}{\epsilon_r} \left( -\frac{\partial h_\phi}{\partial z} \right)$$

$$\frac{\partial e_z}{\partial t} = \frac{1}{\epsilon_z} \frac{1}{r} \frac{\partial}{\partial r} (r \cdot h_\phi) + j_z$$

To accomplish the alignment of the grid's diagonals to the structure's axis the local basis  $\vec{u}$  and  $\vec{v}$  is rotated by an angle of  $45^\circ$  compared to the global basis  $(\vec{r}, \vec{z})$ . Therefore the magnetic field component remains unchanged but both of the electric field components have

to be described with regard to the local basis  $(\vec{u}, \vec{v})$ . This can be seen in the sketch shown in figure 2.

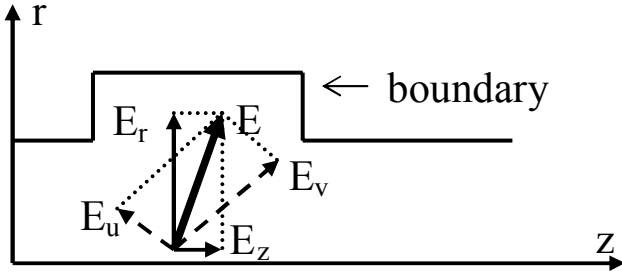


Figure 2: Rotation of coordinates.

By applying the transformation to Maxwell Equations the following set of equations is obtained

$$\begin{aligned}\partial_t h_\phi &= -\frac{1}{\mu_\phi} (\partial_v e_u - \partial_u e_v) \\ \partial_t e_u &= \frac{1}{\epsilon_u} \frac{1}{r} (-\partial_v (r \cdot h_\phi)) + j_u \\ \partial_t e_v &= \frac{1}{\epsilon_v} \frac{1}{r} (\partial_u (r \cdot h_\phi)) + j_v.\end{aligned}$$

The next step is to introduce the exciting fields of the bunch [4]. All field components have to be substituted with the sum of the scattered and the exciting field, i.e.  $total = (s)cattered + (e)xciting$ . The set of equations in the new local basis reads

$$\begin{aligned}\partial_t h_\phi^s &= -\frac{1}{\mu_\phi} (\partial_v e_u^s - \partial_u e_v^s) - \left(1 - \frac{\mu_0}{\mu_\phi}\right) \partial_t h_\phi^e \\ \partial_t e_u^s &= \frac{1}{\epsilon_u} \frac{1}{r} (-\partial_v (r \cdot h_\phi^s)) - \left(1 - \frac{\epsilon_0}{\epsilon_u}\right) \partial_t e_u^e \\ \partial_t e_v^s &= \frac{1}{\epsilon_v} \frac{1}{r} (\partial_u (r \cdot h_\phi^s)) - \left(1 - \frac{\epsilon_0}{\epsilon_v}\right) \partial_t e_v^e.\end{aligned}$$

Now they can be discretised on the mesh to obtain via the leapfrog algorithm an update scheme for the field components. Referring to figure 1 the update scheme for the field components reads

$$\begin{aligned}h_\phi^s|_{i,j}^{n+1} &= h_\phi^s|_{i,j}^n - \frac{\Delta t}{\Delta h} \frac{1}{\mu_\phi|_{i,j}} (e_u^s|_{i,j}^{n+0.5} - e_u^s|_{i+1,j}^{n-0.5} \\ &\quad - (e_v^s|_{i,j}^{n+0.5} - e_v^s|_{i-1,j}^{n-0.5})) - \left(1 - \frac{\mu_0}{\mu_\phi|_{i,j}}\right) (h_\phi^e|_{i,j}^{n+1} - h_\phi^e|_{i,j}^n) \\ e_u^s|_{i-1,j}^{n+0.5} &= e_u^s|_{i-1,j}^{n-0.5} - \frac{\Delta t}{\Delta h} \frac{1}{\epsilon_u|_{i-1,j}} \frac{1}{r|_{i-1,j}} \left(-r|_{i,j} \cdot h_\phi^s|_{i,j}^n \right. \\ &\quad \left. - r|_{i-1,j} \cdot h_\phi^s|_{i-1,j}^n\right) - \left(1 - \frac{\epsilon_0}{\epsilon_u|_{i-1,j}}\right) (e_u^e|_{i-1,j}^{n+0.5} - e_u^e|_{i-1,j}^{n-0.5})\end{aligned}$$

$$\begin{aligned}e_v^s|_{i,j}^{n+0.5} &= e_v^s|_{i,j}^{n-0.5} - \frac{\Delta t}{\Delta h} \frac{1}{\epsilon_v|_{i,j}} \frac{1}{r|_{i,j}} \left(-r|_{i-1,j+1} \cdot h_\phi^s|_{i-1,j+1}^n \right. \\ &\quad \left. - r|_{i,j} \cdot h_\phi^s|_{i,j}^n\right) - \left(1 - \frac{\epsilon_0}{\epsilon_v|_{i,j}}\right) (e_v^e|_{i,j}^{n+0.5} - e_v^e|_{i,j}^{n-0.5})\end{aligned}$$

## RESULTS

Three codes (MAFIA, ECHO and the new one with the rotated mesh) were applied to a long structure consisting of 20 Tesla like cells. It has an overall length of 2.4m. The relativistic Gaussian bunch traverses through this structure. Its RMS width  $\sigma$  is equal to 1mm.

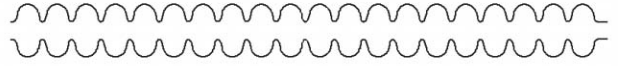


Figure 3: Structure of 20 Tesla cells

Due to the length and the diameter of the structure dispersion effects would occur in a conventional FDTD approach unless a very high spatial mesh resolution is used. This effect can be seen with the MAFIA code. Two simulations were run using two different mesh resolutions. In the first one the spatial resolution is five mesh points on  $\sigma$ . The second one's resolution is twice as good. It is ten mesh points on  $\sigma$ . Both results are compared to the ECHO code which is an implicit code and handles the dispersion problem very well.

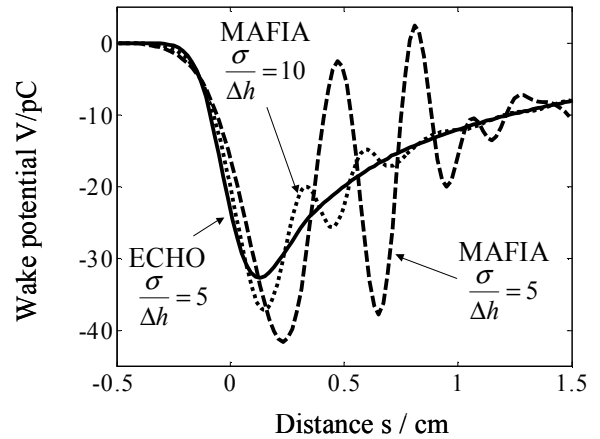


Figure 4: Comparison of wake potentials computed with MAFIA (two different mesh resolutions) and ECHO

Both of the wake potentials computed with the MAFIA code do show oscillations due to dispersion error effects. With increasing mesh resolution i.e. five to ten mesh points per sigma the amplitude of the oscillations is decreasing. Convergence can be observed. As expected the curves converge towards the solid line in figure 4 representing a wake potential computed with the ECHO code.

In principle it is possible to use MAFIA for the case of a long structure and a short bunch. But it demands a very fine mesh, whose density depends linearly on the square root of the structure's length and also is inverse proportional to the square root of the third power of sigma. With the following condition the dispersion error can be suppressed

$$\Delta h \leq \sqrt{\frac{\sigma^3}{L}} \quad (1)$$

Much more better for this purpose is the explicit algorithm introduced in our paper combined with a moving mesh approach. Using the code with the rotated mesh the wake potential for the same structure and the same RMS bunch width was calculated. The mesh resolution is 5 points on  $\sigma$  again.

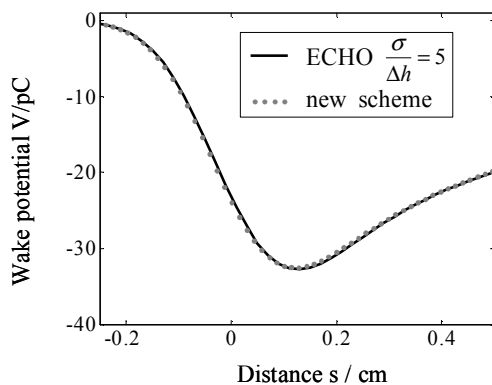


Figure 5: Comparison of wake potentials computed with the ECHO code and the new scheme -  $\sigma=1\text{mm}$

The calculated wake potential of the bunch matches the reference wake potential computed with the ECHO code. Small deviations due to the staircase approximation of the boundary can be observed. To show the performance for shorter bunches the width of the gaussian bunch is reduced by a factor of ten. The structure remains the same.

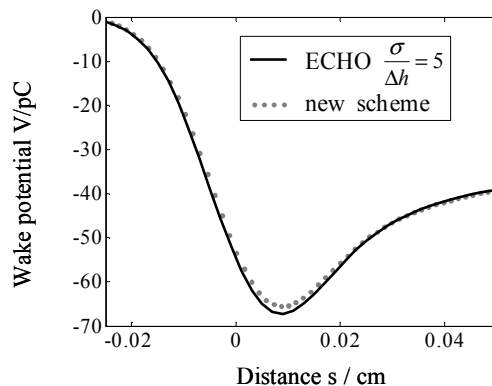


Figure 6: Comparison of wake potentials computed with the ECHO code and the new scheme -  $\sigma=0.1\text{mm}$

Again the new code produces accurate results only with 5 mesh steps per  $\sigma$ . From relation (1) follows that the conventional FDTD approach should use about 160 points on  $\sigma$ .

Another quantity to compare the codes is the loss factor [5]. In the case of five mesh points per sigma and a sigma of 1mm MAFIA gives  $L=17.7372$ , ECHO  $L=21.034$ , and the rotated mesh gives  $L=21.329$  which is quite good compared to ECHO. When decreasing sigma to 0.1mm the loss factors are  $L=46.972$  (ECHO) and  $L=45.549$  (rotated mesh)

## CONCLUSIONS

In this paper we have presented a new discretisation scheme which leads to an explicit approach. It combines the advantage of explicit algorithms with the absence of dispersion in the beam direction.

By a simple rotation of the mesh, it is possible to avoid the dispersion problem – at least in the direction of propagation.

This method enables us to compute the longitudinal wake potential without dispersion error accumulation.

The results show that MAFIA demands a very fine mesh for short bunches traversing through a long accelerator structure. On the other hand very accurate results can be obtained by simple mesh rotation.

With this discretisation scheme and a moving mesh approach, calculations of wake potentials of short bunches in long accelerator structures are possible.

## REFERENCES

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