A COMPARISON OF COSY DA MAPS WITH ANALYTIC FORMULAE FOR ORBIT FUNCTIONS OF A NON-SCALING FFAG ACCELERATOR

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Abstract

Fixed Field Alternating Gradient (FFAG) magnetic lattices with fixed, possibly high, radio-frequency proposed for muon acceleration have unusual requirements: relative momentum swing $\Delta p/p$ of $\pm 30\%$. It is not evident whether the existing accelerator optical design codes are sufficiently accurate for such a large momentum range. It is of particular importance to the non-scaling designs that relative spread of revolution periods be kept below $< 10^{-3}$. Analytic expressions for orbit displacements, tunes and path length have been derived for thick-element models of doublet, F0D0 and FDF triplet lattices; it is this paper's purpose to compare these with values computed by the differential algebra (DA) tool COSY. The mutual agreement of results from independent sources will serve to validate them both.

MODEL AND FORMULAE

Trbojevic *et al*[1] compared analytic formulae for a single cyclotron sector against lattice tools. Here that investigation is extended to the simple lattices considered for non-scaling FFAGs[3]. Analytic formulae for the closed orbits, path length, tunes and betatron functions have been obtained; these can serve as basis for verification of computer codes. As a mathematical necessity, the derivation assumes that elements are either sectoral combined-function magnets or parallel-faced quadrupoles; and both with the entrance and exit faces perpendicular to the reference trajectory. The method depends on the fact that for each momentum there is an arc-of-circle orbit through the sector(s) which is an exact solution of the equations of motion; and that an expansion may be made about each of these to match the orbit to the remainder of the lattice. The calculation has two components: (i) find the local arc orbits; and (ii) solve a matrix equation for the global closed orbit vector. Momentum is retained as a free variable throughout.

Local arc Orbits

Let momentum $p \equiv m_0\gamma_u u$, and bending radius be ρ. Further let subscripts c, u denote reference and general trajectories, respectively. The magnet field is $B_z(\rho) = B_0 + B_1(\rho - \rho_c)$ where B_0, B_1 are the dipole and quadrupole components, respectively. Equating momentum to bending, gives $(p - p_c) = (\rho - \rho_c)(B_0 + B_1\rho)$ with solution:

$$\rho_u \approx \rho_c + (p - p_c)/\mathcal{B} - B_1(p - p_c)^2/\mathcal{B}^3 + \ldots \quad (1)$$

* TRIUMF receives federal funding under a contribution agreement with the National Research Council of Canada.

$\mathcal{B} \equiv (B_0 + B_1\rho_c)$ determines whether the radii increase or decrease with momentum. Here ρ_c is the bending radius at p_c. If $\mathcal{B} < 0$, then arcs of higher momenta are at smaller radii than those of lower momenta; if $\mathcal{B} > 0$, the situation is reversed. For good momentum compaction $|\mathcal{B}|$ should be as large as possible, i.e. $|B_1\rho_c| \gg |B_0|$. The formulae can be generalized to nonlinear field index.

Let $\mathbf{r}, \mathbf{u}, \mathbf{B}$ be position, momentum and field vectors, and $\Delta\mathbf{r}, \Delta\mathbf{u}, \Delta\mathbf{B}$ be their increments. If one expands about the family of reference orbits, one obtains the equation:

$$\gamma_u d\Delta\mathbf{u}/dt = (q/m_0)[\Delta\mathbf{u} \wedge \mathbf{B} + (\mathbf{u} + \Delta\mathbf{u}) \wedge \Delta\mathbf{B}] \ . \quad (2)$$

Linear ODEs result if one drops the second order term $\Delta\mathbf{u} \wedge \Delta\mathbf{B}$. The relative fractional error incurred in this truncation is of order x/ρ_c, that is closed orbit offset x divided by magnet bending radius. The computer tool COSY, retains the second order term and is in principle more exact. COSY differs also in the treatment of the magnetic field; it recognizes that the combined function magnet will also have a body sextupole[4] component of order x/ρ_c. The COSY[5] calculations will be considered to be exact.

Global Closed Orbits

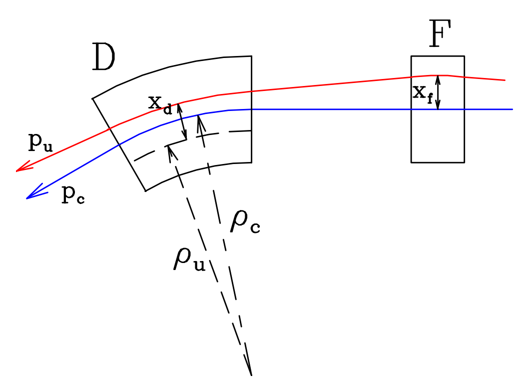

Figure 1: Layout of F0D0 cell with D-sector and F-quad.

We illustrate the method by the simple example of a F0D0 cell with D-sector and F-quad, and introduce the quantities: l_0 is the drift length; l_f, l_d are F-quad and D-sector half-lengths; $k = \sqrt{B_1 c/p}$ where B_1 is the gradient; $\sigma = k \times l$. We transform the input vector $\mathbf{x}_0 = (x_0, 0)$ from the entrance of the half F quadrupole, to the exit of the half D sector using the matrices $\mathbf{D}, \mathbf{F}, \mathbf{O}$. Notice that at entrance to the D sector, the radial coordinate jumps by an amount $\delta\mathbf{r} = (\delta\rho, 0)$ where $\delta\rho = (\rho_u - \rho_c)$ – because of difference between p_u and p_c coordinate systems. Hence the matrix equation:

$$(x, x') = \mathbf{D}_x(k_r l_d)\,[-\delta\mathbf{r} + \mathbf{O}(l_0) \cdot \mathbf{F}_x(k_f l_f) \cdot \mathbf{x}_0] \ . \quad (3)$$

The value of x_0 that will make $x' = 0$ is the *closed orbit*. From this value, the displacement and divergence may be obtained at any other point in the cell by the appropriate matrix multiplication. At the centres of the full F and full D quadrupoles we find the displacements:

$$\begin{aligned} x_f &= (\rho_c - \rho_u)\mu_r \sinh\sigma_r / D && \text{w.r.t. } \rho_c \quad (4)\\ x_d &= (\rho_c - \rho_u) k_f \rho_c \omega_u \sin\sigma_f / D && \text{w.r.t. } \rho_u \quad (5)\\ D &= k_f \rho_c \omega_u \cosh\sigma_r \sin\sigma_f \\ &- \mu_r \sinh\sigma_r (\cos\sigma_f - k_f l_0 \sin\sigma_f)\,. \end{aligned}$$

The working is increasingly more complex for triplet and doublet cells, and particularly so when both the D and F are combined function magnets. Nevertheless, the analgous expressions have been found, and from these path length was obtained by Pythagoras' theorem and integration.

MUON LATTICES

We demonstrate the results of this technique using the example of a 10-20 GeV/c muon accelerator proposed as part of a US neutrino factory[2]. Parameters for F0D0 (F), doublet (D) and triplet (T) lattices are given below; cases with a single D-sector (1) and with D- and F-sectors (2) are considered. In all cases the long/short drift-space is 2/0.5 m. 7.5 MeV of (peak) acceleration is installed in every cell; the radio-frequency is 200 MHz.

Type	F1	F2	D1	D2	T1	T2
#	96	95	90	88	77	81
L	5.70	5.70	4.80	4.73	5.80	5.75
C	547	541	432	416	447	466
l_d	1.14	1.14	1.52	1.45	1.76	1.73
l_f	.561	.563	.784	.781	.519	.512
B_{0d}	3.53	3.97	2.92	3.72	3.04	3.69
B_{1d}	-21.1	-21.2	-22.3	-23.6	-19.9	-20.6
B_{0f}	0	-1.40	0	-1.73	0	-2.09
B_{1f}	40.7	40.5	40.3	41.1	38.1	38.9
ρ_d	17.4	14.2	21.7	15.2	21.6	14.8
ρ_f	–	-40.2	–	-32.9	–	-26.1
p_c	18.4	16.9	19.0	17.0	19.7	16.4

Units in the table are # cells, cell length L (m), circumference C (m), magnet *full* lengths $l_{d,f}$ (m), fields $B_{0d,f}$ (T), gradients $B_{1d,f}$ (T/m), radii $\rho_{d,f}$ (m), p_c (GeV/c).

In figures 3,4,5,6,8,9 the solid curves indicate values computed analytically, while the dashed curves denote values computed from an 8th order COSY map.

For cells with a D-sector and F-quad and $p_c \approx 19$ GeV/c, the maximum anticipated errors are of order 0.5%. The linearized expansion is valid over a wider momentum range than the machines intended operation as shown in figure 3.

For cells with D- and F- sectors and $p_c \approx 16$ GeV/c the reference momentum is better centred (i.e. closer to the mean of 15 GeV/c). Normally the more symmetric expansion would result in greater accuracy. However, the reverse field in the F-sector and the stronger field in the D-sector to compensate implies a smaller radius of curvature and so larger relative errors. Thus, overall, the accuracy over the operational range 10-20 GeV/c is similar to that above. Fig. 4 summarizes the pathlength comparison.

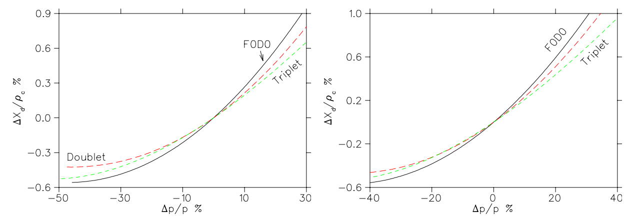

Figure 2: Expected errors for muon lattices F1,D1,T1 (left) and F2,D2,T2 (right).

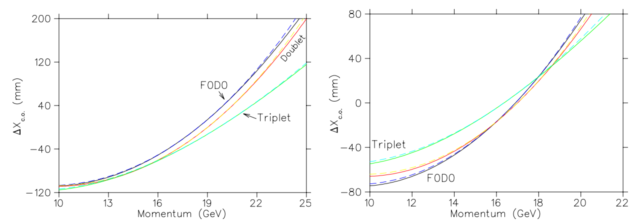

Figure 3: Orbit offsets in F-quad for lattices F1,D1,T1 (left), and in F-sector for lattices F2,D2,T2 (right)

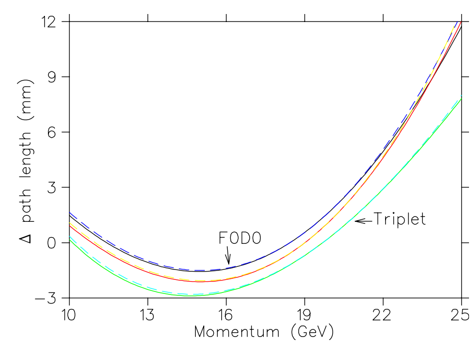

Figure 4: Pathlength variation in muon lattices F1,D1,T1

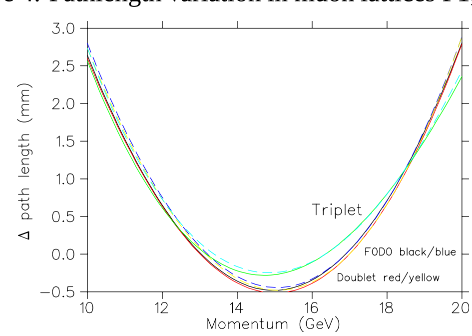

Figure 5: Pathlength variation in muon lattices F2,D2,T2

To summarize, for the high energy muon lattices with small orbit offsets and large magnet bending radii, the analytic results and the COSY DA maps are in excellent agreement upon the values of orbit offset and pathlength. Figures 6 show this agreement extends to the values of the betatron tunes. The solid/dashed curves denote analytic/COSY values; they are almost indistinguishable.

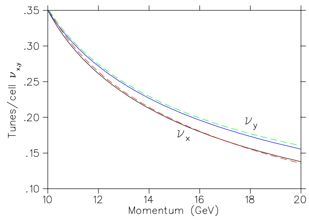

Figure 6: Tunes in muon lattices D2 (left) and T2 (right)

ELECTRON MODEL LATTICES

The non-scaling lattices have two novel operational features: fast crossing of betatron resonances and asynchronous acceleration relying on libration. It is proposed to investigate these features in a 10–20 MeV/c electron model. This would have long/short drift spaces of 10/5 cm, and utilise 2.86 GHz RF with 0.25 MV of acceleration per cell. Example parameters are given below.

Type	F1	F2	D1	D2	T1	T2
#	36	34	33	32	29	28
L	40	40	33	33	41	41
C	14.4	13.6	10.9	10.6	11.9	11.5
l_d	6.63	6.66	8.51	8.44	10.0	10.2
l_f	3.37	3.34	4.49	4.56	2.99	2.91
B_{0d}	.158	.178	.139	.163	.138	.161
B_{1d}	-5.23	-5.34	-5.60	-5.77	-4.77	-4.94
B_{0f}	0	-.062	0	-.072	0	-.092
B_{1f}	9.49	9.52	9.75	9.57	8.973	9.07
ρ_d	38.0	29.7	44.7	32.8	46.3	30.5
ρ_f	–	-85.3	–	-74.5	–	-53.3
p_c	18.0	15.9	18.6	16.0	19.2	14.7

Units in the table are # cells, L (cm), C (m), full length $l_{d,f}$ (cm), $B_{0d,f}$ (T), $B_{1d,f}$ (T/m), $\rho_{d,f}$ (cm), p_c (MeV/c).

The lattices have a small number of cells and large bend angles in the magnets, and the ratio of the closed orbit offsets to the bending radiii are comparatively large. These are conditions which challenge all latice/optics design tools. For example, the COSY maps do not converge over the entire expansion range $(\pm 40\%)$ unless maps of order higher than 6th are used; and if fewer than $\simeq 25$ cells are used, then maps higher than 7th order are required. Figure 7 shows the anticipated relative fractional errors, based on x/ρ_c, to be of order 5% or more at the momentum extremes. Figure 8 shows there to be large discrepancy between COSY and the analytic pathlength formulae; and the same is true of the betatron tunes and lattice functions.

CONCLUSION

Using *Mathematica*[6], analytic formulae for the closed orbits, path length, tunes and betatron functions have been obtained for the simple lattices considered for non-scaling FFAGs. The relative fractional error is anticipated and confirmed to be of order $\epsilon \equiv x/\rho$. For ϵ sufficiently small, COSY and the formulae are in excellent agreement; e.g. the formulae are eminently suitable for design of the high energy muon rings. However, for rings with large bend angle, such as the electron model, the formulae are forced to operate outside their domain of applicability leading to appreciable errors. Moreover, the same is true of low order COSY maps. This observation gives reason to reiterate that caution is warranted[4, 5] when applying optics design codes to small rings.

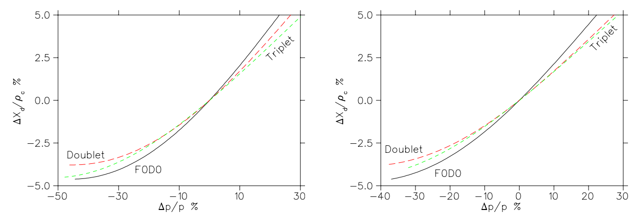

Figure 7: Expected errors in electron lattices F1,D1,T1 (left) and lattices F2,D2,T2 (right).

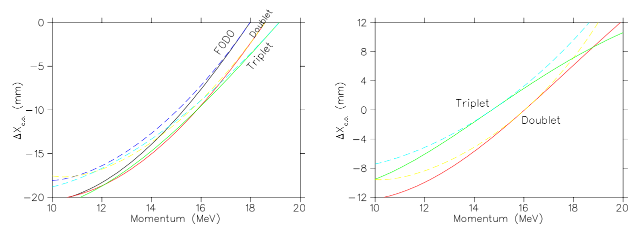

Figure 8: Closed orbit offsets in electron lattices F1,D1,T1 (left) and lattices F2,D2,T2 (right).

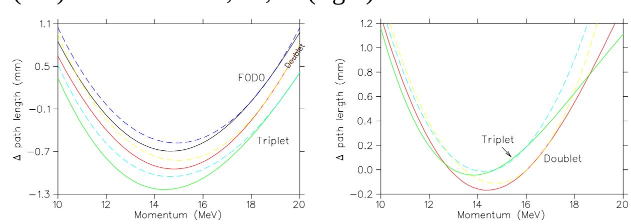

Figure 9: Pathlength variation in electron lattices F1,D1,T1 (left) and F2,D2,T2 (right).

Acknowledgement

The author wishes to thank Dejan Trbojevic of Brookhaven National Laboratory for providing the basic script for extracting closed orbit offsets, pathlengths and tunes, etc., from the COSY maps. Dobrin Kaltchev (TRIUMF) has aided my understanding of COSY and also gave much initial assistance in running the software.

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