TOROIDAL CAVITY LOADED WITH AN ELECTRON BEAM

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Abstract

Three problems have been considered in this paper: the development of Maxwell's equations strict solution method to define the electromagnetic own values and own functions of the toroidal cavity; the radiation of the charged bunch rotating along the average radius, and, at last, the consideration of the case of a toroid filled with dielectric medium. The peculiarities of this radiation have been investigated as well. We suppose to consider further the case when toroid is filled with plasma-like disperse medium.

INTRODUCTION

Two methods to define the own electromagnetic oscillations in toroidal cavity were considered before: the uniform short wave asymptotic method of the own oscillations (USWA) and successive approximate method (SAM) based on the perturbation theory, both for large torus variables are strictly separated in eikonal equation for homogeneous toroidal modes. The first method (USWA) is based on the variables asymptotic separation in Helmholtz equation for toroidal cavity the toroidal inhomogeneous medium filled with. Using this strict solution for scalar fields one may construct an asymptotic solution for electromagnetic ones for large toroid using the method, developed in [1]. The second method (SAM) is based on the perturbation theory, developed in [6], using the small parameter $r/R \ll 1$. Two sets of own frequencies were received using the analytical expressions, received by these two methods. In USWA method we use the classical toroidal system of coordinates (τ, σ, φ), where the surfaces of $\tau = const$ describe toroids, $\sigma = const$ - spheres and $\varphi = const$ - half planes and in SAM - the quasi-spherical one (r, φ, ϑ) [3]. The new original method to find a strict solution is considered as well [4].

USWA METHOD

The USWA method gives the following equation for the E-modes ($E_\varphi \neq 0$)

$$P_{n-1/2}^{i\sqrt{(ka)^2 - m^2}}(\operatorname{ch}\tau) = 0 \text{ at } \tau = \tau_1, \tag{1}$$

where $P_{n-1/2}^{i\sqrt{(ka)^2 - m^2}}(\operatorname{ch}\tau)$ is the torus function [5], $\tau = \tau_1$ - toroid with perfect conducting walls and round cross-section, $k = \omega/c$, where ω is frequency of electromagnetic oscillations in toroid, parameter a describes the toroidal system of coordinates [6] and is connected with the torus radius r_0 and the axial circle radius R defined as $R = a \operatorname{cth}\tau_1$, $r_0 = a/\operatorname{sh}\tau_1$ and $R^2 - r_0^2 = a^2$. The large toroid means that $\tau_1 \gg 1$ and $R \sim a$. It is difficult to define the values for own frequencies from equation (1) and to receive an analytical solution for this equation. But for the large torus ($r/R \ll 1$, or $\tau_1 \gg 1$) we used uniform short wave asymptotics for the torus function ([1]). In this approximation the geometrical-optical beams form a caustic surface with the radius r_c in the torus, which isn't a coaxial one to the main torus $\tau = \tau_1$:

$$r_c = \frac{n}{k}\sqrt{\left(1 - \frac{1}{4n^2}\right) \Big/ \left(1 - \left(\frac{m}{ka}\right)^2\right)} \tag{2}$$

If $ka > m$ this caustic is placed in the torus and if $ka < m$ - is an external one, i.e. for these values no one mode will exist in toroid. We see that the number m is limited by the value ka. This fact simplifies the calculations essentially.

Dielectric filling in the torus is described here by the expression $n = \sqrt{\varepsilon} = \dfrac{\operatorname{ch}\tau - \cos\sigma}{\operatorname{sh}\tau}$, $\mu = 1$. (3)

One may see that if $\tau \to \infty$, then $n \to 1$, i.e. (3) describes an empty large torus.

The synchrotron radiation field generated by rotating charged particle may be defined expanding the current

$$j_\varphi(\tau, \sigma, \varphi) = ev\frac{(\operatorname{ch}\tau - \cos\sigma)^3}{a^3 \operatorname{sh}\tau_0} \times \delta(\tau - \tau_0)\delta(\sigma - \sigma_0)\delta(\varphi - \varphi_0) \tag{4}$$

by the own functions of the toroid [1]

$$E_\varphi = -C\left(1 - \frac{m^2}{k^2 a^2}\right)^{1/2} \sin\frac{\sigma}{2}(\operatorname{ch}\tau - \cos\sigma)^{1/2} \times P_{n-1/2}^{i\sqrt{(ka)^2 - m^2}}(\operatorname{ch}\tau)\, C_{2n-1}^{(1)}(\cos\sigma/2) \begin{matrix}\cos m\varphi \\ \sin m\varphi\end{matrix} \tag{5}$$

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for the E-types of oscillations ($H_\varphi = 0$), where $C_{2n-1}^{(1)}(\cos\sigma/2)$ - ultra-spherical functions (the polynoms of Gegenbauer) of $2n-1$ order.

We can approximate the torus function for $ka >> 1$ in this form:

$$P_{n-1/2}^{i\sqrt{(ka)^2-m^2}}(\operatorname{ch}\tau) \approx \left[\sqrt{\beta(\tau)}\operatorname{sh}\tau\right]^{1/2} \times c\cos\left(k\int_{\tilde\tau}^{\tau}\sqrt{\beta(\tau)}d\tau - \pi/4\right) \quad (6)$$

We have shown that it is possible to define the own frequencies of toroidal cavity for given values of numbers n and m. At first for given n, one can define the coordinates of caustic surface from then replacing the latter in caustic equation one can define the k wave numbers according the values of m:

$$ka = \sqrt{m^2 + \left(n^2 - 1/4\right)\operatorname{sh}^2\tilde\tau}\,. \quad (7)$$

One can see that during increasing of the value n the electromagnetic oscillations will approach to the walls of toroid.

THE POSSIBILITY OF STRICT SOLUTION AND SAM METHOD

We represent the electromagnetic fields $\mathbf{E},\mathbf{H}(r,\varphi,\vartheta)$ in the form

$$\mathbf{E},\mathbf{H}(r,\varphi,\vartheta) = \mathbf{E}(r,\vartheta),\mathbf{H}(r,\vartheta)\begin{matrix}\cos m\varphi\\ \sin m\varphi\end{matrix}, \quad (8)$$

in quasi-spherical system of coordinates (r,φ,ϑ) [4]. Putting these fields in Maxwell's equations one can express all components of fields by $E_\varphi(r,\vartheta)$, i.e. one can show that E- and H- types of waves exist in the toroidal cavities. The component $E_\varphi(r,\vartheta)$ satisfies the equation

$$\nabla^2_{r,\vartheta}E_\varphi(r,\vartheta) + \left\{k^2 - \frac{m^2}{R^2h^2}\right\}E_\varphi(r,\vartheta) = 0, \quad (9)$$

where

$$\nabla^2_{r,\vartheta}E_\varphi(r,\vartheta) = \frac{\partial^2 E_\varphi}{\partial r^2} + \frac{1-\rho_0\cos\vartheta}{rh}\frac{\partial E_\varphi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 E_\varphi}{\partial\vartheta^2} + \frac{\sin\vartheta}{rR(1-2\rho_0\cos\vartheta)}\frac{\partial E_\varphi}{\partial\vartheta}. \quad (9a)$$

Solving this equation by digital methods one can define the own electromagnetic oscillations in the toroidal cavity:

$$\begin{aligned}
E_r &= -\frac{mR\,tgm\varphi}{(khR)^2 + m^2tg^2m\varphi}\frac{\partial}{\partial r}\left[hE_\varphi\right],\\
E_\vartheta &= -\frac{mR\,tgm\varphi}{(khR)^2 + m^2tg^2m\varphi}\frac{1}{r}\frac{\partial}{\partial\varphi}\left[hE_\varphi\right],\\
E_\varphi(r,\varphi,\vartheta) &= E_\varphi(r,\vartheta)\begin{matrix}\cos m\varphi\\ \sin m\varphi\end{matrix},\\
H_r &= -\frac{ikhR^2}{(khR)^2 + m^2tg^2m\varphi}\frac{1}{r}\frac{\partial}{\partial r}\left[hE_\varphi\right],\\
H_\vartheta &= \frac{ikhR^2}{(khR)^2 + m^2tg^2m\varphi}\frac{\partial}{\partial r}\left[hE_\varphi\right],\\
H_\varphi &= 0,
\end{aligned} \quad (10)$$

for the E-types of waves. One can get similar equations for the H-types of waves [4]. In these equations $h = 1 - \rho_0\cos\vartheta$.

Let's use another approximate SAM method to define the own oscillations in the quasi-spherical system of coordinates (r,φ,ϑ). The own frequencies are defined now in the form

$$\omega_{nm} = \frac{c}{r_0}\sqrt{\chi_{on}^2 + \left(m^2 + 0.75\right)\rho_0^2}\,. \quad (11)$$

where $\rho_0 = \frac{r_0}{R}$, r_0 is the radius of the cross-sections of toroidal cavity and R is the average radius of the torus, χ_{0n} - is the n-th root of the zero order Bessel function and m - azimuthal number. The first order correction stipulated by the torus curvature exists in the formula (11) in the form $\left(m^2 + 0.75\right)\rho_0^2$.

The package FEMLAB gives possibility to solve the problem of eigen values and eigen functions digitally by finite element method (FEM). The toroidal cavity is represented here like a figure rotating round the axis Oz at the distance R. The wave equation is solved here in the cylindrical system of coordinates. When it is solved by the Dirichlet's boundary conditions it is corresponding to the uniform modes of E-types. Let's try to get the own frequencies for the toroid with parameters $r_0 = 3cm$ and $r_0 = 5cm$ and $R = 10cm$ and compare with the frequencies defined by (11).

$m/3$	f_{1m}^{FEM}	f_{1m}^{SAM}
1	3.87803 GHz	3.8794 GHz
2	3.9675 GHz	3.9666 GHz
3	4.11183 GHz	4.1078 GHz
4	4.3048 GHz	4.2976 GHz
5	4.53936 GHz	4.5301 GHz
6	4.80838 GHz	4.7989 GHz

$m/5$	f_{1m}^{FEM}	f_{1m}^{SAM}
1	2.3872 GHz	2.3819 GHz
2	2.53512 GHz	2.5214 GHz
3	2.75795 GHz	2.7381 GHz
4	3.03216 GHz	3.0155 GHz
5	3.33821 GHz	3.3384 GHz
6	5.66252 GHz	3.6949 GHz

In SAM the radius of the caustic surface is defined as

$$r_c = \frac{n}{k}. \tag{12}$$

Comparing (12) with (2) we see that the difference between these two values is neglectible, if $n >> 1$, $m << ka$. SAM method permits one to calculate the approximate values of the field in torus when radiating particle is moving along the axis of the toroid, existing there for a finite time and rotating in it at α angle. At $t \le \alpha/\Omega$ the transition radiation will be written down in the form of the expansion in series

$$E_\varphi^{tr} = -\frac{4ec\left(\beta^2\varepsilon - 1\right)}{\pi a^2 R\beta\sqrt{h}\varepsilon^2} \times \sum_p\sum_m \frac{\varepsilon_m J_0\left(\chi_{0p}r\right)}{J_1^2\left(\chi_{0p}\right)} \frac{\omega_{pm}\sin\left(\omega_{pm}t\right)}{m^2\Omega^2 - \omega_{pm}^2}\cos m\varphi\,, \tag{13}$$

and the synchrotron radiation-in the form

$$E_\varphi^{sc} = \frac{4ec\left(\beta^2\varepsilon - 1\right)}{\pi a^2 R\beta\sqrt{h}\varepsilon^2} \times \sum_p\sum_m \frac{\varepsilon_m J_0\left(\chi_{0p}r\right)}{J_1^2\left(\chi_{0p}\right)} \frac{m\Omega\sin\left(m\Omega t\right)}{m^2\Omega^2 - \omega_{pm}^2}\cos m\varphi\,. \tag{14}$$

In fig. 1 there are shown the results of the calculation of radiation field E_φ per unit charge in the empty toroid ($\varepsilon = 1$). As one can see, the transition radiation is prevailing for the case of $K = 10$, $\rho_0 = 0.243$, $m = 1$.

Thus, these formulae describe the transition radiation (effect of sudden acceleration) as well. Particle arises and makes some circles in toroid (angle α) and then vanishes.

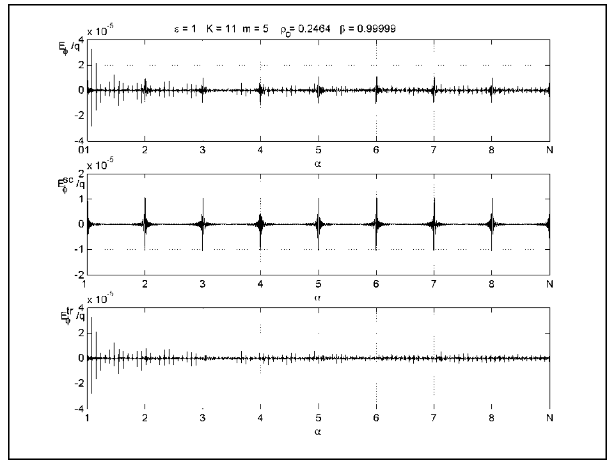

Figure 1: Field E_φ/q and its components E_φ^{tr}/q and E_φ^{sc}/q corresponding to transition and synchrotron radiations (8 circles in empty toroid).

CONCLUSION

Combining and comparing the results of methods USWA and SAM from one side and with method FEMLAB from other side which gives a possibility to define eigen frequencies digitally, one can talk about the completeness of the set of own functions defined asymptotically. The strict method developed in this work gives possibility to define the own frequencies of toroidal cavity independently. The coincidence of these results will prove the completeness of the set of own functions defined asymptotically as well.

REFERENCES

[1] E.D. Gazazyan. “Uniform Short Wave Asymptotics of Scalar and Electromagnetic Fields Based on the One-dimensional Standard Functions,” Preprint YerPhI-1092(55)88, Yerevan, 1988.

[2] E.D. Gazazyan, V.G. Kocharyan, G.G. Oxuzyan, “Toroidal Cavities with the Rectangular and Circle Cross Section,” Preprint YerPhI-1145(22)89, Yerevan, 1989.

[3] G. Korn, T. Korn, Mathematical Handbook, McGraw-Hill Book Company, 1968.

[4] T.A. Harutyunyan, D.K. Kalantaryan, Electromagnetic Oscillations in the Toroidal Cavity, Contemporary Physics Armenian Ac. Sc. (in print).

[5] M. Abramovits, I.A. Stegun, Handbook of Mathematical Functions, National Bureau of Standards, Applied Mathematics Series-55, June 1964.

[6] T. Liley, B. Shnizer, R. Kiel, “Perturbation Theory Computation of Toroidal Uniform Modes within Empty Torus,” AEU, 1983, vol. 37, p. 359-365.