A SIMPLIFIED MODEL OF INTRABEAM SCATTERING

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Abstract

Beginning with the general Bjorken-Mtingwa solution, we derive a simplified model of intrabeam scattering (IBS), one valid for high energy beams in normal storage rings; our result is similar, though more accurate than a model due to Raubenheimer. In addition, we show that a modified version of Piwinski's IBS formulation (where $\eta^2_{x,y}/\beta_{x,y}$ has been replaced by $\mathcal{H}_{x,y}$) at high energies asymptotically approaches the same result.

1 INTRODUCTION

Intrabeam scattering (IBS), an effect that tends to increase the beam emittance, is important in hadronic[1] and heavy ion[2] circular machines, as well as in low emittance electron storage rings[3]. In the former type of machines it results in emittances that continually increase with time; in the latter type, in steady-state emittances that are larger than those given by quantum excitation/synchrotron radiation alone.

The theory of intrabeam scattering for accelerators was first developed by Piwinski[4], a result that was extended by Martini[5], to give a formulation that we call here the standard Piwinski (P) method[6]; this was followed by the equally detailed Bjorken and Mtingwa (B-M) result[7]. Both approaches solve the local, two-particle Coulomb scattering problem for (six-dimensional) Gaussian, uncoupled beams, but the two results appear to be different; of the two, the B-M result is thought to be the more general[8].

For both the P and the B-M methods solving for the IBS growth rates is time consuming, involving, at each time (or iteration) step, a numerical integration at every lattice element. Therefore, simpler, more approximate formulations of IBS have been developed over the years: there are approximate solutions of Parzen[9], Le Duff[10], Raubenheimer[11], and Wei[12]. In the present report we derive—starting with the general B-M formalism—another approximation, one valid for high energy beams and more accurate than Raubenheimer's approximation. We, in addition, demonstrate that under these same conditions a modified version of Piwinski's IBS formulation asymptotically becomes equal to this result.

2 HIGH ENERGY APPROXIMATION

2.1 The General B-M Solution[7]

Let us consider bunched beams that are uncoupled, and include vertical dispersion due to *e.g.* orbit errors. Let the intrabeam scattering growth rates be

$$\frac{1}{T_p}=\frac{1}{\sigma_p}\frac{d\sigma_p}{dt}\ ,\quad \frac{1}{T_x}=\frac{1}{\epsilon_x^{1/2}}\frac{d\epsilon_x^{1/2}}{dt}\ ,\quad \frac{1}{T_y}=\frac{1}{\epsilon_y^{1/2}}\frac{d\epsilon_y^{1/2}}{dt}\ , \tag{1}$$

with σ_p the relative energy spread, ϵ_x the horizontal emittance, and ϵ_y the vertical emittance. The growth rates according to Bjorken-Mtingwa (including a $\sqrt{2}$ correction factor[13], and including vertical dispersion) are

$$\frac{1}{T_i} = 4\pi A(\log)\left\langle \int_0^\infty \frac{d\lambda\,\lambda^{1/2}}{[\det(L+\lambda I)]^{1/2}}\left\{ TrL^{(i)}Tr\left(\frac{1}{L+\lambda I}\right) - 3TrL^{(i)}\left(\frac{1}{L+\lambda I}\right)\right\}\right\rangle \tag{2}$$

where i represents p, x, or y;

$$A=\frac{r_0^2cN}{64\pi^2\bar\beta^3\gamma^4\epsilon_x\epsilon_y\sigma_s\sigma_p}\quad , \tag{3}$$

with r_0 the classical particle radius, c the speed of light, N the bunch population, $\bar\beta$ the velocity over c, γ the Lorentz energy factor, and σ_s the bunch length; $(\log)$ represents the Coulomb log factor, $\langle\rangle$ means that the enclosed quantities, combinations of beam parameters and lattice properties, are averaged around the entire ring; $\det$ and Tr signify, respectively, the determinant and the trace of a matrix, and I is the unit matrix. Auxiliary matrices are defined as

$$L=L^{(p)}+L^{(x)}+L^{(y)}\quad , \tag{4}$$

$$L^{(p)}=\frac{\gamma^2}{\sigma_p^2}\begin{pmatrix}0&0&0\\0&1&0\\0&0&0\end{pmatrix}\quad , \tag{5}$$

$$L^{(x)}=\frac{\beta_x}{\epsilon_x}\begin{pmatrix}1&-\gamma\phi_x&0\\-\gamma\phi_x&\gamma^2\mathcal{H}_x/\beta_x&0\\0&0&0\end{pmatrix}\quad , \tag{6}$$

$$L^{(y)}=\frac{\beta_y}{\epsilon_y}\begin{pmatrix}0&0&0\\0&\gamma^2\mathcal{H}_y/\beta_y&-\gamma\phi_y\\0&-\gamma\phi_y&1\end{pmatrix}\quad . \tag{7}$$

The dispersion invariant is $\mathcal{H}=[\eta^2+(\beta\eta'-\frac{1}{2}\beta'\eta)^2]/\beta$, and $\phi=\eta'-\frac{1}{2}\beta'\eta/\beta$, where β and η are the beta and dispersion lattice functions.

The Bjorken-Mtingwa Solution at High Energies

Let us first consider $1/T_p$ as given by Eq. 2. Note that if we change the integration variable to $\lambda'=\lambda\sigma_H^2/\gamma^2$ then

$$(L+\lambda' I)=\frac{\gamma^2}{\sigma_H^2}\begin{pmatrix}a^2+\lambda'&-a\zeta_x&0\\-a\zeta_x&1+\lambda'&-b\zeta_y\\0&-b\zeta_y&b^2+\lambda'\end{pmatrix}\quad , \tag{8}$$

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with

$$\frac{1}{\sigma_H^2} = \frac{1}{\sigma_p^2} + \frac{\mathcal{H}_x}{\epsilon_x} + \frac{\mathcal{H}_y}{\epsilon_y} \quad , \tag{9}$$

$$a = \frac{\sigma_H}{\gamma}\sqrt{\frac{\beta_x}{\epsilon_x}} , \quad b = \frac{\sigma_H}{\gamma}\sqrt{\frac{\beta_y}{\epsilon_y}} , \quad \zeta_{x,y} = \phi_{x,y}\sigma_H\sqrt{\frac{\beta_{x,y}}{\epsilon_{x,y}}} \tag{10}$$

Note that, other than a multiplicative factor, there are only 4 parameters in this matrix: a, b, ζ_x, ζ_x. Note that, since $\beta\phi^2 \le \mathcal{H}$, the parameters $\zeta < 1$; and that if $\mathcal{H} \approx \eta^2/\beta$ then ζ is small. We give, in Table 1, average values of a, b, ζ_x, in selected electron rings.

Table 1: Average values of a, b, ζ_x, in selected electron rings. The zero current emittance ratio $\sim 0.5\%$ in all cases.

Machine	E[GeV]	$N[10^{10}]$	$\langle a\rangle$	$\langle b\rangle$	$\langle\zeta_x\rangle$
KEK's ATF	1.4	.9	.01	.10	.15
NLC	2.0	.75	.01	.20	.40
ALS	1.0	5.	.015	.25	.15

Let us limit consideration to high energies, specifically let us assume a,$b \ll 1$ (if the beam is cooler longitudinally than transversely, then this is satisfied). We note that all 3 rings in Table 1, on average, satisfy this condition reasonably well. Assuming this condition, the 2nd term in the braces of Eq. 2 is small compared to the first term, and we drop it. Our second assumption is to drop off-diagonal terms (let $\zeta = 0$), and then all matrices will be diagonal.

Simplifying the remaining integral by applying the high energy assumption we finally obtain

$$\frac{1}{T_p} \approx \frac{r_0^2 cN(\log)}{16\gamma^3\epsilon_x^{3/4}\epsilon_y^{3/4}\sigma_s\sigma_p^3}\left\langle \sigma_H\, g(a/b)\, (\beta_x\beta_y)^{-1/4}\right\rangle , \tag{11}$$

with

$$g(\alpha) = \frac{2\sqrt{\alpha}}{\pi}\int_0^\infty \frac{du}{\sqrt{1+u^2}\sqrt{\alpha^2+u^2}} \quad . \tag{12}$$

A plot of $g(\alpha)$ over the interval $[0 < \alpha < 1]$ is given in Fig. 1; to obtain the results for $\alpha > 1$, note that $g(\alpha) = g(1/\alpha)$. A fit to g,

$$g(\alpha) \approx \alpha^{(0.021-0.044\ln\alpha)} \qquad [\text{for } 0.01 < \alpha < 1] \quad , \tag{13}$$

is given by the dashes in Fig. 1. The fit has a maximum error of 1.5% over $[0.02 \le \alpha \le 1]$.

Similarly, beginning with the 2nd and 3rd of Eqs. 2, we obtain

$$\frac{1}{T_{x,y}} \approx \frac{\sigma_p^2\langle\mathcal{H}_{x,y}\rangle}{\epsilon_{x,y}}\frac{1}{T_p} \quad . \tag{14}$$

Our approximate IBS solution is Eqs. 11,14. Note that Parzen's high energy formula is a similar, though more approximate, result to that given here[9]; and Raubenheimer's approximation is Eq. 11, with $g(a/b)\sigma_H/\sigma_p$ replaced by $\frac{1}{2}$, and Eqs. 14 exactly as given here[11].

Note that the beam properties in Eqs. 11,14, need to be the self-consistent values. Thus, for example, to find the

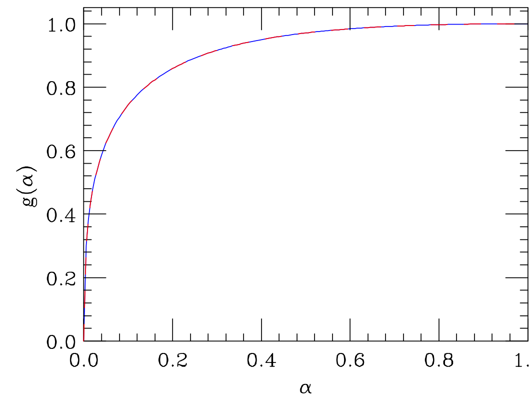

Figure 1: The auxiliary function $g(\alpha)$ (solid curve) and the approximation, $g = \alpha^{(0.021-0.044\ln\alpha)}$ (dashes).

steady-state growth rates in electron machines, iteration will be required[6]. Note also that these equations assume that the zero-current vertical emittance is due mainly to vertical dispersion caused by orbit errors; if it is due mainly to (weak) x-y coupling we let $\mathcal{H}_y = 0$, drop the $1/T_y$ equation, and let $\epsilon_y = \kappa\epsilon_x$, with κ the coupling factor[3].

What sort of error does our model produce? Consider a position in the ring where $\zeta_y = 0$. In Fig. 2 we plot the ratio of the *local* growth rate T_p^{-1} as given by our model to that given by Eq. 2 as function of ζ_x, for example combinations of a and b. We see that for $\zeta_x \lesssim \sqrt{b}e^{(1-\sqrt{b})}$ (which is typically true in storage rings) the dependance on ζ_x is weak and can be ignored. In this region we see that the model approaches B-M from above as a,$b \to 0$. Finally, adding small $\zeta_y \neq 0$ will reduce slightly the ratio of Fig. 2.

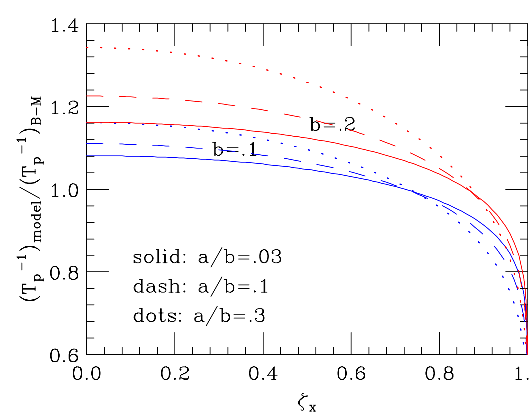

Figure 2: The ratio of *local* growth rates in p as function of ζ_x, for $b = 0.1$ (blue) and $b = 0.2$ (red) $[\zeta_y = 0]$.

3 COMPARISON TO PIWINSKI

3.1 *The Standard Piwinski Solution[6]*

The standard Piwinski solution is

$$\frac{1}{T_p} = A\left\langle \frac{\sigma_h^2}{\sigma_p^2} f(\tilde a, \tilde b, q)\right\rangle$$

$$\frac{1}{T_x} = A\left\langle f(\frac{1}{\tilde a}, \frac{\tilde b}{\tilde a}, \frac{q}{\tilde a}) + \frac{\eta_x^2\sigma_h^2}{\beta_x\epsilon_x} f(\tilde a, \tilde b, q)\right\rangle$$

$$\frac{1}{T_y} \quad = \quad A \left\langle f(\frac{1}{\tilde{b}}, \frac{\tilde{a}}{\tilde{b}}, \frac{q}{\tilde{b}}) + \frac{\eta_y^2 \sigma_h^2}{\beta_y \epsilon_y} f(\tilde{a}, \tilde{b}, q) \right\rangle \; ; \quad (15)$$

$$\frac{1}{\sigma_h^2} = \frac{1}{\sigma_p^2} + \frac{\eta_x^2}{\beta_x \epsilon_x} + \frac{\eta_y^2}{\beta_y \epsilon_y} \quad , \quad (16)$$

$$\tilde{a} = \frac{\sigma_h}{\gamma}\sqrt{\frac{\beta_x}{\epsilon_x}}, \quad \tilde{b} = \frac{\sigma_h}{\gamma}\sqrt{\frac{\beta_y}{\epsilon_y}}, \quad q = \sigma_h \beta \sqrt{\frac{2d}{r_0}} \quad ; \quad (17)$$

the function f is given by:

$$f(\tilde{a}, \tilde{b}, q) \quad = \quad 8\pi \int_0^1 du \, \frac{1 - 3u^2}{PQ} \; \times$$
$$\times \quad \left\{ 2 \ln \left[\frac{q}{2} \left(\frac{1}{P} + \frac{1}{Q} \right) \right] - 0.577 \ldots \right\} (18)$$

$$P^2 = \tilde{a}^2 + (1 - \tilde{a}^2)u^2, \qquad Q^2 = \tilde{b}^2 + (1 - \tilde{b}^2)u^2 \; . \quad (19)$$

The parameter d functions as a maximum impact parameter, and is normally taken as the vertical beam size.

3.2 Comparison of Modified Piwinski to the B-M Solution at High Energies

We note that Piwinski's result depends on η^2/β, and not on $\mathcal{H}$ and ϕ, as the B-M result does. This may suffice for rings with $\langle \mathcal{H} \rangle \approx \langle \eta^2/\beta \rangle$. For a general comparison, however, let us consider a formulation that we call the *modified* Piwinski solution. It is the standard version of Piwinski, but with η^2/β replaced by $\mathcal{H}$ (*i.e.* $\tilde{a}$, $\tilde{b}$, σ_h, become a, b, σ_H, respectively).

Let us consider high energy beams, *i.e.* let a,$b \ll 1$: First, notice that in the integral of the auxiliary function f (Eq. 18): the -0.577 can be replaced by 0; the $-3u^2$ in the numerator can be set to 0; P (Q) can be replaced by $\sqrt{a^2 + u^2}$ ($\sqrt{b^2 + u^2}$). The first term in the braces can be approximated by a constant and then be pulled out of the integral; it becomes the effective Coulomb log factor. Note that for the proper choice of the Piwinski parameter d, the effective Coulomb log can be made the same as the B-M parameter $(\log)$. For flat beams ($a \ll b$), the Coulomb log of Piwinski becomes $(\log) = \ln\left[d\sigma_H^2/(4r_0 a^2)\right]$.

We finally obtain, for the first of Eqs. 15,

$$\frac{1}{T_p} \approx \frac{r_0^2 c N (\log)}{16 \gamma^3 \epsilon_x^{3/4} \epsilon_y^{3/4} \sigma_s \sigma_p^3} \left\langle \sigma_H \, h(a,b) \; (\beta_x \beta_y)^{-1/4} \right\rangle \; , \quad (20)$$

with

$$h(a,b) = \frac{2\sqrt{ab}}{\pi} \int_0^1 \frac{du}{\sqrt{a^2 + u^2}\sqrt{b^2 + u^2}} \quad . \quad (21)$$

We see that the the approximate equation for $1/T_p$ for high energy beams according to modified Piwinski is the same as that for B-M, except that $h(a,b)$ replaces $g(a/b)$. But for a,b small, $h(a,b) \approx g(a/b)$, and the Piwinski result approaches the B-M result. For example, for the ATF with $\epsilon_y/\epsilon_x \sim 0.01$, $a \sim 0.01$, $a/b \sim 0.1$, and $h(a,b)/g(a/b) = 0.97$; the agreement is quite good.

Finally, for the relation between the transverse to longitudinal growth rates according to modified Piwinski: note that for non-zero vertical dispersion the second term in the brackets of Eqs. 15 (but with $\eta_{x,y}^2/\beta_{x,y}$ replaced by $\mathcal{H}_{x,y}$), will tend to dominate over the first term, and the results become the same as for the B-M method.

In summary, we have shown that for high energy beams (a,$b \ll 1$), in normal rings (ζ not very close to 1): if the parameter d in P is chosen to give the same equivalent Coulomb log as in B-M, then the *modified* Piwinski solution agrees with the Bjorken-Mtingwa solution.

4 NUMERICAL COMPARISON[3]

We consider a numerical comparison between results of the general B-M method, the modified Piwinski method, and Eqs. 11,14. The example is the ATF ring with no coupling; to generate vertical errors, magnets were randomly offset by 15 μm, and the closed orbit was found. For this example $\langle \mathcal{H}_y \rangle = 17\ \mu$m, yielding a zero-current emittance ratio of 0.7%; the beam current is 3.1 mA. The steady-state growth rates according to the 3 methods are given in Table 2. We note that the Piwinski results are 4.5% low, and the results of Eqs. 11,14, agree very well with those of B-M. Additionally, note that, not only the (averaged) growth rates, but even the *local* growth rates around the ring agree well for the three cases. Finally, note that for coupling dominated NLC, ALS examples ($\kappa = 0.5\%$, see Table 1) the error in the steady-state growth rates (T_p^{-1},T_x^{-1}) obtained with the model is (12%,2%), (7%,0%), respectively.

Table 2: Steady-state IBS growth rates (in [s^{-1}]) for an ATF example with vertical dispersion due to random errors.

Method	$1/T_p$	$1/T_x$	$1/T_y$
Modified Piwinski	25.9	24.7	18.5
Bjorken-Mtingwa	27.0	26.0	19.4
Eqs. 11,14	27.4	26.0	19.4

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