NONLINEAR LOCALIZED COHERENT SPECTRUM OF BEAM-BEAM INTERACTIONS

Antonina N. Fedorova, Michael G. Zeitlin
IPME, RAS, St. Petersburg, V.O. Bolshoj pr., 61, 199178, Russia *†

Abstract

We consider modeling for strong-strong beam-beam interactions beyond preceding linearized/perturbative methods such as soft gaussian approximation or FMM (HFMM) etc. In our approach discrete coherent modes, discovered before, and possible incoherent oscillations appear as a result of multiresolution/multiscale fast convergent decomposition in the bases of high-localized exact nonlinear modes represented by wavelets or wavelet packets functions. The constructed solutions represent the full multiscale spectrum in all internal hidden scales from slow to fast oscillating eigenmodes. Underlying variational method provides algebraical control of the spectrum.

1 INTRODUCTION

We consider the first steps of analysis of beam-beam interactions in some collective model approach. It is well-known that neither direct PIC modeling nor soft-gaussian approximation provide reasonable resolution of computing time/noise problems and understanding of underlying complex nonlinear dynamics [1], [2]. Recent analysis, based as on numerical simulation as on modeling, demonstrates that presence of coherent modes inside the spectrum leads to oscillations and growth of beam transverse size and deformations of beam shape. This leads to strong limitations for operation of LHC. Additional problems appear as a result of continuum spectrum of incoherent oscillations in each beam. The strong-strong collisions of two beams also lead to variation of transverse size. According to [2] it is reasonable to find nonperturbative solutions at least in the important particular cases. Our approach based on wavelet analysis technique is in some sense the direct generalization of Fast Multipole Method (FMM) and related approaches (HFMM). After set-up based on Vlasov-like model (according [2]) in part 2, we consider variational-wavelet approach [3]-[17] in framework of powerful technique based on Fast Wavelet Transform (FWT) operator representations [18] in section 3. As a result we have multiresolution/multiscale fast convergent decomposition in the bases of high-localized exact nonlinear eigenmodes represented by wavelets or wavelet packets functions. The constructed solutions represent the full multiscale spectrum in all internal hidden scales from slow to fast oscillating eigenmodes. Underlying variational method provides algebraical control of the spectrum.

* e-mail: zeitlin@math.ipme.ru
† http://www.ipme.ru/zeitlin.html; http://www.ipme.nw.ru/zeitlin.html

2 VLASOV MODEL FOR BEAM-BEAM INTERACTIONS

Vlasov-like equations describing evolution of the phase space distributions $\psi^j = \psi^j(x, p_x, \theta)$ $(j = 1, 2)$ for each beam are [2]:

$$\frac{\partial \psi^j}{\partial \theta} = -q_x p_x \frac{\partial \psi^j}{\partial x} + \Big(q_x x + \delta_p(\theta) 4\pi \xi_x p.v. \int_{-\infty}^{\infty} \frac{\rho^*(x', \theta)}{x - x'} \mathrm{d}x'\Big) \frac{\partial \psi^j}{\partial p_x} \tag{1}$$

where

$$\rho^*(x, \theta) = \int_{-\infty}^{\infty} \psi^*(x, p_x, \theta) \mathrm{d}p_x \tag{2}$$

and ψ^* is the density of the opposite beam, q_x is unperturbed fractional tune, ξ_x is horizontal beam-beam parameter, N is a number of particles, x, p_x are normalized variables. This model describes horizontal oscillations of flat beams with one bunch per beam, one interaction point, equal energy, population and optics for both beams.

3 FWT BASED VARIATIONAL APPROACH

One of the key points of wavelet approach demonstrates that for a large class of operators wavelets are good approximation for true eigenvectors and the corresponding matrices are almost diagonal. FWT [18] gives the maximum sparse form of operators under consideration (1). It is true also in case of our Vlasov-like system of equations (1). We have both differential and integral operators inside. So, let us denote our (integral/differential) operator from equations (1) as T $(L^2(R^n) \to L^2(R^n))$ and its kernel as K. We have the following representation:

$$< Tf, g >= \int \int K(x, y) f(y) g(x) \mathrm{d}x \mathrm{d}y \tag{3}$$

In case when f and g are wavelets $\varphi_{j,k} = 2^{j/2}\varphi(2^j x - k)$ (3) provides the standard representation for operator T. Let us consider multiresolution representation $\ldots \subset V_2 \subset V_1 \subset V_0 \subset V_{-1} \subset V_{-2} \ldots$. The basis in each V_j is $\varphi_{j,k}(x)$, where indices k, j represent translations and scaling respectively. Let $P_j : L^2(R^n) \to V_j$ $(j \in Z)$ be projection operators on the subspace V_j corresponding to level j of resolution: $(P_j f)(x) = \sum_k < f, \varphi_{j,k} > \varphi_{j,k}(x)$. Let $Q_j = P_{j-1} - P_j$ be the projection operator on the subspace W_j $(V_{j-1} = V_j \oplus W_j)$ then we have the following "microscopic or telescopic" representation of operator T which

takes into account contributions from each level of resolution from different scales starting with the coarsest and ending to the finest scales [18]:

$$T = \sum_{j \in Z} (Q_j T Q_j + Q_j T P_j + P_j T Q_j). \qquad (4)$$

We remember that this is a result of presence of affine group inside this construction. The non-standard form of operator representation [18] is a representation of operator T as a chain of triples $T = \{A_j, B_j, \Gamma_j\}_{j \in Z}$, acting on the subspaces V_j and W_j: $A_j : W_j \to W_j, B_j : V_j \to W_j, \Gamma_j : W_j \to V_j$, where operators $\{A_j, B_j, \Gamma_j\}_{j \in Z}$ are defined as $A_j = Q_j T Q_j, \quad B_j = Q_j T P_j, \quad \Gamma_j = P_j T Q_j$. The operator T admits a recursive definition via

$$T_j = \begin{pmatrix} A_{j+1} & B_{j+1} \\ \Gamma_{j+1} & T_{j+1} \end{pmatrix}, \qquad (5)$$

where $T_j = P_j T P_j$ and T_j acts on $V_j : V_j \to V_j$. It should be noted that operator A_j describes interaction on the scale j independently from other scales, operators B_j, Γ_j describe interaction between the scale j and all coarser scales, the operator T_j is an "averaged" version of T_{j-1}. We may compute such non-standard representations for different operators (including Calderon-Zygmund or pseudodifferential). As in case of differential operator as in other cases we need only to solve the system of linear algebraical equations. The action of integral operator in equations (1) we may consider as a Hilbert transform

$$(H\rho^*)(x) = \frac{1}{\pi} p.v. \int_{-\infty}^{\infty} \frac{\rho^*(x', \theta)}{x' - x} \mathrm{d}x' \qquad (6)$$

The representation of H on V_0 is defined by the coefficients

$$r_\ell = \int \varphi(x - \ell)(H\varphi)(x) \mathrm{d}x, \quad \ell \in Z. \qquad (7)$$

which according to FWT technique define also all other coefficients of the nonstandard representation. So we have $H = \{A_j, B_j, \Gamma_j\}_{j \in Z}$ with the corresponding matrix elements $a_{i-\ell}, b_{i-\ell}, c_{i-\ell}$ which can be computed from coefficients r_ℓ only:

$$\begin{aligned} a_i &= \sum_{k,k'=0}^{L-1} g_k g_{k'} r_{2i+k-k'} \\ b_i &= \sum_{k,k'=0}^{L-1} g_k h_{k'} r_{2i+k-k'} \\ c_i &= \sum_{k,k'=0}^{L-1} h_k g_{k'} r_{2i+k-k'} \end{aligned} \qquad (8)$$

The coefficients r_ℓ (7) can be obtained from

$$r_\ell = r_{2\ell} + \sum_{k=1}^{L/2} d_{2k-1} (r_{2\ell-2k+1} + r_{2\ell+2k-1}) \qquad (9)$$

where d_n are the so called autocorrelation coefficients of the corresponding quadratic mirror filter $\{h_k\}_{k=0}^{L-1}$: $d_n = 2 \sum_{i=0}^{L-1-n} h_i h_{i+n}$, $n = 1, \ldots, L-1$, $d_{2k} = 0$, $k = 1, \ldots, L/2-1$, $g_k = (-1)^k h_{L-k-1}$, $k = 0, \ldots, L-1$, which parametrizes the basic refinement equation $\varphi(x) = \sqrt{2} \sum_{k=0}^{L-1} h_k \varphi(2x - k)$. This equation really generates all wavelet zoo. It is useful to add to the system (9) the following asymptotic condition $r_\ell = -1/\pi\ell + O(\ell^{-2M})$, which simplifies the solution procedure. Then finally we have the following action of operator T_j on sufficiently smooth function f:

$$(T_j f)(x) = \sum_{k \in Z} \left(2^{-j} \sum_\ell r_\ell f_{j,k-\ell} \right) \varphi_{j,k}(x), \qquad (10)$$

in the wavelet basis $\varphi_{j,k}(x) = 2^{-j/2} \varphi(2^{-j}x - k)$ where

$$f_{j,k-1} = 2^{-j/2} \int f(x) \varphi(2^{-j}x - k + \ell) \mathrm{d}x \qquad (11)$$

are wavelet coefficients. So, we have simple linear parametrization of matrix representation of our operator (6) in wavelet bases and of the action of this operator on arbitrary vector in proper functional space. The similar approach can be applied to other operators in (1). Then we may apply our variational approach from [3]-[17]. Let L be an arbitrary (non) linear (differential/integral) operator corresponds to the system (1) with matrix dimension d, which acts on some set of functions $\Psi \equiv \Psi(\theta, x, p_x) = \left(\Psi^1(\theta, x, p_x), \ldots, \Psi^d(\theta, x, p_x) \right), \quad \theta, x, p_x \in \Omega \subset \mathbf{R}^3$, $L\Psi \equiv L(Q, \theta, x, p_x)\Psi(\theta, x, p_x) = 0$, where $Q \equiv Q_{d_1,d_2,d_3}(\theta, x, p_x, \partial/\partial\theta, \partial/\partial x, \partial/\partial p_x, \int \mathrm{d}x \mathrm{d}p_x)$. Let us consider now the N mode approximation for solution as the following ansatz (in the same way we may consider different ansatzes) [17]:

$$\Psi^N(\theta, x, p_x) = \sum_{r,s,k=1}^{N} a_{rsk} A_r \otimes B_s \otimes C_k(\theta, x, p_x) \qquad (12)$$

We shall determine the coefficients of expansion from the following conditions (different related variational approaches are considered in [3]-[16]):

$$\ell^N_{k\ell m} \equiv \int (L\Psi^N) A_k(\theta) B_\ell(x) C_m(p_x) \mathrm{d}\theta \mathrm{d}x \mathrm{d}p_x = 0$$

So, we have exactly dN^3 algebraical equations for dN^3 unknowns a_{rsk}. The solution is parametrized by solutions of two set of reduced algebraical problems, one is linear or nonlinear (depends on the structure of operator L) and the rest are some linear problems related to computation of coefficients of algebraic equations. These coefficients can be found by some wavelet methods by using compactly supported wavelet basis functions for expansions (12). We may consider also different types of wavelets including general wavelet packets. The constructed solu-

tion has the following multiscale/multiresolution decomposition via nonlinear high-localized eigenmodes

$$\psi(\theta, x, p_x) = \sum_{(i,j,k)\in Z^3} a_{ijk} A^i(\theta) B^j(x) C^k(p_x), \quad (13)$$

$$A^i(\theta) = A_N^{i,slow}(\theta) + \sum_{r\geq N} A_r^i(\omega_r \theta),\ \omega_r \sim 2^r$$

$$B^j(x) = B_M^{j,slow}(x) + \sum_{l\geq M} B_l^j(k_l^1 x),\ k_l^1 \sim 2^l$$

$$C^s(p_x) = C_L^{s,slow}(p_x) + \sum_{m\geq L} C_m^s(k_m^2 p_x),\ k_m^2 \sim 2^m$$

which corresponds to the full multiresolution expansion in all underlying time/space scales. Formula (13) gives

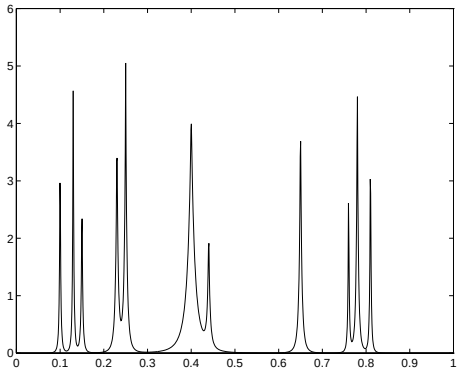

Figure 1: Region of nonlinear resonances.

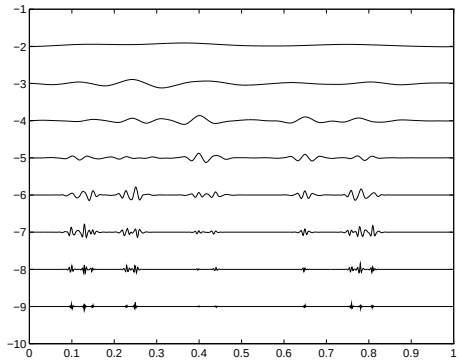

Figure 2: Eigenmodes decomposition.

us expansion into the slow part $f_{N,M,L}^{slow}$ and fast oscillating parts for arbitrary N, M, L. So, we may move from coarse scales of resolution to the finest one to obtain more detailed information about our dynamical process. The first terms in the RHS of formulae (13) correspond on the global level of function space decomposition to resolution space and the second ones to detail space. The using of wavelet basis with high-localized properties provides fast convergence of constructed decomposition (13). In contrast with different approaches, formulae (13) does not use perturbation technique or linearization procedures and represents dynamics via generalized nonlinear localized eigenmodes expansion. Numerical calculations are based on compactly supported wavelets and related wavelet families. Figures 1,2 demonstrate resonances region and corresponding nonlinear coherent eigenmodes decomposition according to representation (13).

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