BEAM DYNAMICS FORMATION IN MAGNETIC FIELD

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Abstract

In this paper methods of the construction of magnetic fields forming given beam dynamics are developed. We consider this problem as the inverse problem of determination of magnetic field by preassigned motion of charged particles. Different inverse problems of electrodynamics have always been the subject of attention of many researchers. Attempts to find approaches to the solving of problems of such kind had been undertaken for a long time. Similar problems had been considered in the works of G.A.Grinberg, A.R.Lucas, B.Meltzer, V.T.Ovcharov, V.I.Zubov [1-5]. In this paper numerical-analytical method of finding magnetic field is developed. The investigated problem is reduced to the solving of the Goursat problem for quasi-linear system of differential equations with partial derivations. To solve this problem the method of characteristics and the Masso method are developed.

1 INTRODUCTION

We consider the following approach to the problem of beam dynamics formation in magnetic field. Let us have the velocities field

$$\begin{cases} \dfrac{dr}{dt} = \eta_r(r,\theta,z), \\ \dfrac{d\theta}{dt} = \dfrac{1}{r}\eta_\theta(r,\theta,z), \\ \dfrac{dz}{dt} = \eta_z(r,\theta,z). \end{cases} \tag{1}$$

Here r,θ,z are cylindrical coordinates, $\eta_r(r,\theta,z)$, $\eta_\theta(r,\theta,z), \eta_z(r,\theta,z)$ are components of the given vector function η in cylindrical system of coordinates, t is time.

Let us describe charged particle motion in stationary magnetic field by the system of differential equations

$$\begin{cases} \dfrac{dr}{dt} = Y_r, \\ \dfrac{d\theta}{dt} = \dfrac{1}{r}Y_\theta, \\ \dfrac{dz}{dt} = Y_z, \\ m\dfrac{dY}{dt} = eY \times B, \end{cases} \tag{2}$$

where we have $Y = (Y_r, Y_\theta, Y_z)$ as particle velocity vector in cylindrical system of coordinates, m is particle mass, e is particle charge, $B = (B_r, B_\theta, B_z)$ is vector function, determining magnetic induction.

It is necessary to find stationary magnetic field (i.e. vector function B which satisfies Maxwell equation $\operatorname{div} B = 0$) in which charged particles motion accords to the velocities field (1). This means that in configuration space charged particles trajectories determined by system (2) under the same initial conditions will coincide with trajectories of the system (1). We have the following theorem [6].

Theorem 1. *Let η be given twice continuously differentiable vector function, and $\eta^2 = \text{const}$. Then such magnetic field exists in which charged particles motions defined by the system (2) coincides with motions of the system (1), when initial conditions of these motions are the same. This magnetic field B can be represented in the form*

$$B = -\frac{m}{e}\ \operatorname{rot}\ \eta + h\eta,$$

where $h = h(r,\theta,z)$ is an arbitrary function satisfying the following condition

$$\operatorname{div}\ (h\eta) = 0\ .$$

2 PROBLEM STATEMENT

In this paper we consider in details the problem of charged particles dynamics formation in the case of stationary axially symmetric magnetic field, i.e. we suppose that $B_\theta = 0,\ B_r = B_r(r,z),\ B_z = B_z(r,z)$. In many problems of focusing and transportation of charged particles we are interested in particles behavior in the plane (r,z). Therefore, we consider the following statement of the problem. Let us suppose, velocities field is given, as

$$\frac{dr}{dz} = f(r,z), \tag{3}$$

i.e. $f(r,z)$ is a given function, which provides the needed motion of charged particles in the plane (r,z).

It is needed to find stationary axially symmetric magnetic field in which the charged particles motions describe by the system (2) coincide with the motions determined by the equation (3) in the plane (r,z).

Let us consider the following system of differential equations [6]:

$$\begin{cases} -fl\dfrac{\partial L}{\partial z} + l\dfrac{\partial L}{\partial r} = d_1, \\ -q\dfrac{\partial h}{\partial z} + g\dfrac{\partial L}{\partial z} - fq\dfrac{\partial h}{\partial r} + fg\dfrac{\partial L}{\partial r} = d_2. \end{cases} \tag{4}$$

Here $q = c_1^2 - L^2/r^2$, $g = hL/r^2$, $l = \frac{m}{e} L/r^2$,

$d_1 = \frac{m}{e} L^2/r^3 - \frac{m}{e} Fq + rg\left(q(1+f^2)\right)^{1/2}$,

$F = \left(\frac{\partial f}{\partial z} + \frac{\partial f}{\partial r} f\right) \Big/ \left(1+f^2\right)$,

$d_2 = h\left(c_1^2 \frac{f}{r} + q\left(\frac{\partial f}{\partial r} - fF\right)\right)$, c_1 is a total velocity of particle.

We assume, the solution of the system (5) will be twice continuously differentiable vector function $L = L(r,z)$ and continuously differentiable vector function $h = h(r,z)$ making this equations in identities. We can formulate the following theorem [6].

Theorem 2. *For arbitrary continuously differentiable function f, for which the solution of the system (4) exists, there is such axially symmetric magnetic field B, where charged particle motions determined by system (2) coincides in the plane (r,z) with the motions of equation (3). The considered motions had of course the same initial conditions. The components of magnetic field B are determined by the formulae*

$$B_r = \frac{m}{er}\frac{\partial L}{\partial z} + fh\left(\frac{c_1^2 - L^2/r^2}{1+f^2}\right)^{1/2},$$
$$B_z = -\frac{m}{er}\frac{\partial L}{\partial r} + h\left(\frac{c_1^2 - L^2/r^2}{1+f^2}\right)^{1/2}. \tag{5}$$

Further we will consider the algorithm of numerical finding of these components.

3 THE CHARACTERISTIC EQUATIONS

Thus, in the case of axially symmetric stationary magnetic field the problem of charged particle dynamics formation is reduced to solving system (4). It is the system of quasi–linear partial differential equations of the first order. The system (4) depends on some arbitrary function $f(r,z)$ determining the velocities field in the plane (r,z). Under any choice of this function the system (4) will be the hyperbolic system. We consider the way of solving this system on the base of characteristics method.

Let us write out for it the equations of characteristic directions and differential correlation for them.

The first family:

$$dr = f dz$$
$$q dh + g dL - d_2 dz = 0 \tag{6}$$

The second family:

$$dz = -f dr$$
$$l dL - d_1 dr = 0 \tag{7}$$

Characteristics directions equations are determined just by the function $f(r,z)$ and they do not depend on unknown functions.

4 GOURSAT PROBLEM

Let us consider the problem of finding solution for the system (4). For numerical solving of this problem it is necessary to assign additional conditions. Using differential correlations (6), (7) for the system (4) we will consider the Goursat problem, i.e. we will search for the solutions $L(r,z), h(r,z)$ of the system (4) under the condition that on two characteristics, outgoing from one point, values of L and h are given.

We will assume that function $f(r,z)$, defining velocities field is such that $f(0,z) = f(r,0) = 0$. It is not difficult to notice that in such case straight line $r = 0$ is characteristic for the first family and the straight line $z = 0$ for the second one.

Let us formulate Goursat problem in the following way. To find the solution of the system (6) assuming that function L and h are given on characteristics $r = 0$ and $z = 0$

$$L(0,z) = L_{r0}(z),\ h(0,z) = h_{r0}(z),\ 0 \le z \le z_n,$$
$$L(r,0) = L_{z0}(r),\ h(r,0) = h_{z0}(r),\ 0 \le r \le r_m,$$

here $L_{r0}(z)$, $L_{z0}(r)$, $h_{r0}(z)$, $h_{z0}(r)$ are given functions, z_n, r_m are given values. In additional, values of corresponding functions, given on characteristics, coincide in the generic point

$$L_{r0}(0) = L_{z0}(0),\ \ h_{r0}(0) = h_{z0}(0).$$

Naturally, it is supposed, that given functions $L_{r0}(z), L_{z0}(r), h_{r0}(z), h_{z0}(r)$ on every characteristics are such that differential characteristics equations satisfied. Further we examine one of possible ways of L and h functions assignment on the corresponding characteristics. Let $L_{r0} = 0$, $L_{z0}(r) = -kr^2$, here k is some constant. We determine the values of function h on mentioned above characteristics, i.e. functions $h_{r0}(z), h_{z0}(r)$, using corresponding differential equations on considered characteristics. As result we get

$h_{z0}(r) = -\frac{m}{e} k\left(c_1^2 - k^2 r^2\right)^{-1/2} - \frac{m}{ekr}\frac{\partial f}{\partial z}(r,0)\left(c_1^2 - k^2 r^2\right)^{1/2}$,

and by solving differential equation

$$dh_{r0} + h_{r0}\left(\frac{f}{r}(0,z) + \frac{\partial f}{\partial r}(0,z)\right)dz = 0$$

with initial condition $h_{r0} = -\frac{mk}{ec_1} - \frac{mc_1}{ek}\left(\frac{\partial f}{\partial z}\Big/ r\right)(0,0)$ we will find the function $h_{r0}(z)$.

5 MASSO METHOD

For the construction of numerical solution of Goursat problem we will use Masso method, based on the substitution of differential equations of characteristics for corresponding finite difference equations.

Let us consider equations of characteristic directions, obtained above

$$dr = f dz, \quad (8)$$

$$dz = -f dr. \quad (9)$$

They determine two orthogonal characteristic families. As differential equations of characteristic directions not depend on functions L, h, so at first we can construct network, formed by intersection of characteristic families, i.e. we will find the coordinates of points (i, j) : $r(i, j)$ and $z(i, j)$, $i = 1,...,n$, $j = 1,...,m$. The system for coordinates determination of point (i, j), $i = 1,...,n$, $j = 1,...,m$ is following

$$r(i, j) = (r(i-1, j) + f_1 f_2 r(i, j-1) + f_1 (z(i, j-1) - z(i-1, j)))/(1 + f_1 f_2),$$

$$z(i, j) = (z(i, j-1) + f_1 f_2 z(i-1, j) + f_2 (r(i, j-1) - r(i-1, j)))/(1 + f_1 f_2),$$

where $f_1 = f(r(i-1), z(i-1, j))$ is value of function f in the point $(i-1, j)$, $f_2 = f(r(i, j-1), z(i, j-1))$ is value of function f in the point $(i, j-1)$.

Afterwards we will calculate values of function L and h in nodes of obtained network. We consider the characteristic differential equations (6), (7) for the system (4). By replacing differentials in differential correlations on corresponding characteristics by finite differences we get system of equations for determination of L and h values in point (i, j), when L and h values in points $(i-1, j)$ and $(i, j-1)$ are known

$$L(i, j) = L(i, j-1) + \frac{d_1}{l}(i, j-1)(r(i, j) - r(i, j-1)),$$

$$h(i, j) = h(i-1, j) + \frac{g}{q}(i, j)(L(i, j) - L(i-1, j)) - \frac{d_2}{q}(i-1, j)(z(i, j) - z(i-1, j)).$$

Let us consider formulae (5) for determination of magnetic field components through the functions L and h being the solution of the system. Under numerical solving of the system (4) we will find the numerical values of magnetic fields components. We will find numerical values $B_r(i, j)$, $B_z(i, j)$ by numerical values of $L(i, j)$ and $h(i, j)$ in nodes of the network. Derivative $\frac{dL}{dr}$ along to the solutions of the system (9) and derivative $\frac{dL}{dz}$ along to the solutions of the system (8) have the form

$$\frac{dL}{dr} = \frac{\partial L}{\partial r} - f \frac{\partial L}{\partial z},$$

$$\frac{dL}{dz} = \frac{\partial L}{\partial z} + f \frac{\partial L}{\partial r}.$$

Hence we can find partial derivatives $\frac{\partial L}{\partial r}$ and $\frac{\partial L}{\partial z}$, as such

$$\frac{\partial L}{\partial r} = \left(\frac{dL}{dr} + f \frac{dL}{dz}\right) \Big/ \left(1 + f^2\right), \quad \frac{\partial L}{\partial z} = \left(\frac{dL}{dz} - f \frac{dL}{dr}\right) \Big/ \left(1 + f^2\right), \quad (10)$$

To get numerical meanings of $\frac{\partial L}{\partial r}$ and $\frac{\partial L}{\partial z}$ in nodes of network it is necessary to replace in (11) differentials by corresponding finite differences

$$\frac{dL}{dr}(i, j) = (L(i, j+1) - L(i, j))/(r(i, j+1) - r(i, j)),$$

$$\frac{dL}{dz}(i, j) = (L(i+1, j) - L(i, j))/(z(i+1, j) - z(i, j)).$$

Thus $B_r(i, j)$, $B_z(i, j)$ magnetic field components in the network nodes are determined by formulae (5), where the values of $\frac{\partial L}{\partial r}, \frac{\partial L}{\partial z}, L, h, f, r$ are taken in corresponding points (i, j).

6 CONCLUSION

The proposed method allows to find the magnetic field, initiating the needed motions of charged particles. We shall note that choice of functions $L_{r0}(z), L_{z0}(r)$ contains some difficulty. Although obtained magnetic field will satisfy the main task of providing given dynamics of charged particles, it may, however, be that this magnetic field does not satisfy one or another practical requirement connected with realization of this field. Bearing this in mind, these functions can be considered as control functions. So, we can consider not only the problem of magnetic field finding, providing given dynamics of charged particles, but needed characteristics of magnetic field.

7 REFERENCES

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