ANALYSIS OF FORMULAS USED IN COUPLING IMPEDANCE COAXIAL-WIRE MEASUREMENTS FOR DISTRIBUTED IMPEDANCES*

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Abstract

In this paper we study the validity of coupling impedance bench measurements for distributed impedances, comparing the commonly used log formula to the result obtained applying a modified version of Bethe's theory of diffraction to a long slot in a coaxial beam pipe. The equations found provide a quantitative expression for the influence of the wire thickness used in the measurement of the real and imaginary part of the longitudinal impedance. The precision achievable in an actual measurement is therefore discussed. The method presented has also been applied in the presence of lumped impedances [1].

1 INTRODUCTION

Bethe's diffraction theory, in its modified version [2], has been successfully used to analytically calculate the coupling impedance of different structures that can be found in an accelerator vacuum chamber [3-6].

More recently, several papers have been dedicated to the theory of coupling impedance bench measurements, in particular regarding the classic coaxial wire method [7-8].

In this paper we use Bethe's theory to calculate the longitudinal impedance of a long and narrow slot, of length L, on a coaxial beam pipe, as it would be ideally measured with the coaxial wire experimental set-up. The analytical formula obtained is compared to the formula derived in [6], which has been checked against MAFIA simulations and other semi-analytical methods. This comparison gives some insight on the influence of the wire on the measurement and on the differences between the various formulas used to relate the measured scattering parameters to the actual impedance.

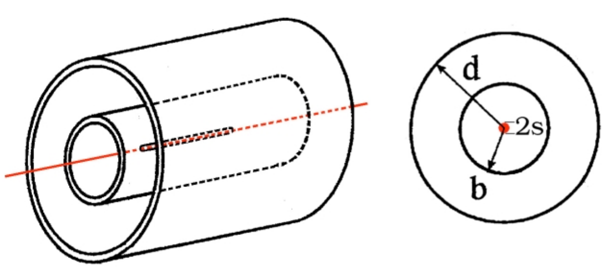

Figure 1: Relevant geometry

2 MEASURED IMPEDANCE

The longitudinal impedance of a long (with respect to the wavelength) slot on a coaxial beam pipe as shown in Fig.1 can be calculated from the measured S_{21} parameter using the Walling (or log) formula [9]:

$$Z_{\log} = -2Z_c \ln\left(S_{21}^{DUT} / S_{21}^{REF}\right) \qquad (1)$$

where Z_c is the characteristic impedance for a coaxial line equal to $60\ \ln(b/s)\ \Omega$ or $Z_0/2\pi$.

In Eq.(1) the S_{21} measured for the device under test is normalized to that of a reference section of equal length. In the following we will always assume that the reference line has been calibrated out and will simply refer to S_{21}.

We also assume perfect matching in the measuring equipment and lossless materials and we will consider only TEM waves which is a rigorous treatment for frequencies below the $TE_{1,1}$ mode cut-off, above which this measurement technique is not accurate anyway.

2.1 Measured Impedance Calculation

In the absence of the coupling aperture, the incident field (E_{0r}, $H_{0\varphi}$) is of course confined to the inner coaxial line and travels the length of the component experiencing only a phase delay. When this delay is taken into account by normalizing with the reference section, S_{21}=1 and the impedance is zero, as expected.

The presence of the aperture generates forward and backward scattered waves travelling in both coaxial regions. From the scattering matrix definition we can write:

$$S_{21} = \frac{H_{0\varphi} + H_{i\varphi}^{+}}{H_{0\varphi}} = \frac{E_{0r} + E_{ir}^{+}}{E_{0r}} \qquad (2)$$

where $(E_{ir}^{+}, H_{i\varphi}^{+})$ is the forward wave in the inner region.

This waves can be expressed as the integral sum of the waves generated by each infinitesimal element of the slot. For the forward scattered wave, we can write:

$$\begin{aligned} E_{ir}^{+}(z > L/2) &= \int_{-L/2}^{L/2} dE_{ir}^{+}(z,z')dz' \\ H_{i\varphi}^{+}(z > L/2) &= \int_{-L/2}^{L/2} dH_{i\varphi}^{+}(z,z')dz' \end{aligned} \qquad (3)$$

The "differential" waves in turn can be written as:

$$\begin{aligned} dE_{ir}^{+}(z,z') &= dc_i^{+}(z')e_{ir}e^{-jk_0(z-z')}\theta(z-z') \\ dH_{i\varphi}^{+}(z,z') &= dc_i^{+}(z')h_{i\varphi}e^{-jk_0(z-z')}\theta(z-z') \end{aligned} \qquad (4)$$

where $k_0=2\pi/\lambda$ is the wavenumber, $\theta(z)$ is the Heaviside function, $Z_0=377\ \Omega$ is the vacuum impedance, e_{ir} and $h_{i\varphi}$ are the TEM modal function in the inner coax:

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$$e_{ir} = \sqrt{\frac{Z_0}{2\pi}} \frac{1}{\sqrt{\ln(b/s)}} \frac{1}{r}$$
$$h_{i\varphi} = \frac{1}{\sqrt{2\pi Z_0}} \frac{1}{\sqrt{\ln(b/s)}} \frac{1}{r} \quad (5)$$

and the excitation constant dc_i^+ depends on the equivalent differential dipole moments of the aperture dM_φ and dP_r:

$$dc_i^+(z') = -\frac{j\omega}{2}\left[\mu h_{i\varphi} dM_\varphi(z') + e_{ir} P_r(z')\right]_{r=b} \quad (6)$$

The equivalent differential dipole moments depend on the aperture polarizabilities α_e and $\alpha_{m\perp}$ and on the incident and scattered fields:

$$dM_\varphi(z) = d\alpha_{m\perp}\left[H_{0\varphi}(z) + H_{i\varphi}(z) - H_{e\varphi}(z)\right]_{r=b}$$
$$dP_r(z) = \varepsilon d\alpha_e\left[E_{0r}(z) + E_{ir}(z) - E_{e\varphi}(z)\right]_{r=b} \quad (7)$$

where (E_{er}, $H_{e\varphi}$) is the scattered field in the outer coaxial region for which equations analogous to Eqs.(3-6) are valid, if we just replace ln*(b/s)* with ln*(d/b)* and the subscript *i* with *e*.

The perturbation technique for calculating the differential dipole moments from the integral equation derived from Eqs.(4-7) is described in [6].

The first order solution yields:

$$\left(\frac{dM_\varphi}{dz}\right)^{1st} = \left(\frac{dM_\varphi}{dz}\right)^{0th} + - j\frac{\omega}{2}\frac{\alpha_m}{L^2}\mu H_{0\varphi}(0)h_{0\varphi}^2(\alpha_m I_1 - \alpha_e I_2)$$
$$\left(\frac{dP_r}{dz}\right)^{1st} = \left(\frac{dP_r}{dz}\right)^{0th} + - j\frac{\omega}{2}\frac{\alpha_e}{L^2}\frac{\mu}{c} H_{0\varphi}(0)h_{0\varphi}^2(\alpha_e I_1 - \alpha_m I_2) \quad (8)$$

where

$$I_1 = \int_{-L/2}^{L/2} e^{-jk_0\xi} e^{-jk_0|z-\xi|} d\xi$$
$$I_2 = \int_{-L/2}^{L/2} sign(\xi - z) e^{-jk_0\xi} e^{-jk_0|z-\xi|} d\xi \quad (9)$$

We can finally calculate the fields in Eq.(3) finding, after some lenghty algebraic passes:

$$H_{i\varphi}^+(z) = -\frac{(k_0 Z_0)^2}{8} H_{0\varphi}(z) h_{i\varphi}^2\left(h_{i\varphi}^2 + h_{e\varphi}^2\right) \cdot \left[(\alpha_e+\alpha_m)^2 + 2(\alpha_e-\alpha_m)^2\frac{1-\cos(2k_0L)}{(2k_0L)^2}\right] + j\frac{k_0 Z_0}{2} H_{0\varphi}(z) h_{i\varphi}^2 \cdot \left[(\alpha_e+\alpha_m) + (\alpha_e-\alpha_m)^2 \cdot \frac{k_0 Z_0}{2}\left(h_{i\varphi}^2 + h_{e\varphi}^2\right)\frac{1}{2k_0L}\left(1 - \frac{\sin(2k_0L)}{2k_0L}\right)\right] \quad (10)$$

We can then calculate S_{21} from Eqs.(2) and (10):

$$S_{21} = 1 + \frac{k_0^2}{32\pi^2 b^4 \ln(b/s)\Lambda}\left[(\alpha_e+\alpha_m)^2 + 2(\alpha_e-\alpha_m)^2\frac{1-\cos(2k_0L)}{(2k_0L)^2}\right] + j\frac{k_0}{4\pi b^4 \ln(b/s)}\left[(\alpha_e+\alpha_m) + \frac{(\alpha_e-\alpha_m)^2}{8\pi b^4 L\Lambda}\left(1 - \frac{\sin(2k_0L)}{2k_0L}\right)\right] \quad (11)$$

where

$$\Lambda = \frac{\ln(d/b)\ln(b/s)}{\ln(d/b) + \ln(b/s)} \quad (12)$$

3 COMPARISON WITH THEORY

An analytical formula for the longitudinal impedance of a long slot at low frequencies has been derived in [6] using the differential modified Bethe's diffraction theory:

$$Z_{//} \approx \frac{Z_0 k_0}{4\pi^2 b^2}\left\{j\left[(\alpha_e+\alpha_m) + -\frac{(\alpha_e-\alpha_m)^2}{8\pi b^2 L \ln(d/b)}\left(1 - \frac{\sin(2k_0L)}{2k_0L}\right)\right] + \frac{k_0}{8\pi b^2 \ln(d/b)}\left[(\alpha_e+\alpha_m)^2 + 2(\alpha_e-\alpha_m)^2\frac{1-\cos(2k_0L)}{(2k_0L)^2}\right]\right\} \quad (13)$$

We can compare Eq.(13) to the longitudinal impedance obtained replacing the S21 value calculated in Eq.(11) into Walling's formula (Eq.(1)). We find:

$$Z_{\log} \approx \frac{k_0 Z_0}{4\pi^2 b^2}\left\{j(\alpha_e+\alpha_{m\perp}) + \frac{k_0}{8\pi b^2 \ln(d/b)}\frac{\ln(d/s)}{\ln(b/s)} \cdot \left[(\alpha_e+\alpha_{m\perp})^2 + 2(\alpha_e-\alpha_{m\perp})^2\frac{1-\cos(2k_0L)}{(2k_0L)^2}\right]\right\} \quad (14)$$

where we have neglected all the higher order terms in α_e, $\alpha_{m\perp}$ and k_o, which corresponds to limiting the analysis to slots of small transverse dimensions ("narrow" slots) and to low frequencies.

3.1 Imaginary Impedance

We can see immediately that the imaginary part of the impedance, as calculated in Eqs.(13) and (14) is exactly the same:

$$\mathrm{Im}(Z_{//}) \approx j\frac{k_0 Z_0}{4\pi^2 b^2}(\alpha_e + \alpha_{m\perp}) \qquad (15)$$

This is not surprising since the imaginary impedance is dominated by the reactive energy stored in the modes below cut-off (i.e. non propagating) near the aperture. These modes are, of course, not much influenced by the presence of the wire, when our approximations are valid.

It is worth pointing out that, apparently, Eq.(17) is totally independent from the wire radius s. This is not so as, if s should increase to become comparable with b, the aperture polarizabilities would be modified.

3.2 Real Impedance

In this case Eqs.(13) and (14) coincide only in the limit $s \rightarrow 0$ and it is well known that, in principle, one would like to use the thinnest possible coaxial wire in the measurements, if it were not for impedance matching, signal-to-noise ratio and mechanical contingent problems. The difference between the two equations is only in that

$$\mathrm{Re}(Z_{\log}) = \mathrm{Re}(Z_{//})\frac{\ln(d/s)}{\ln(b/s)} \qquad (16)$$

This means that it would conceptually be possible to reconstruct the theoretical value of the longitudinal impedance from its value measured using Walling's formula multiplied by the factor $\ln(b/s)/\ln(d/s)$.

Another aspect to notice is that, being obviously always $d>b$, the measured impedance value is always in excess of the theoretical one.

Finally, from a general point of view, we can say that the presence of the coaxial wire supports the propagation of a TEM mode in the inner coaxial region. This would cause the real part of the impedance to be greater than zero even if there were no propagating modes in the outer coaxial region, in which case the theoretical formulas gives zero instead.

4 CONCLUSIONS

In this paper we used a modification of Bethe's diffraction theory, in its differential form, to calculate the impedance of a long slot in a coaxial beam pipe as it would be measured using the classic coaxial wire technique. This result has been compared to the impedance value obtained applying directly the diffraction theory.

The imaginary part of the impedance is not affected, in first approximation, by the wire presence and the standard *log* formula give the same result as the direct calculation.

For the real impedance, the formula gives an impedance always larger than the theoretical value.

It is possible to write a general formula that allows to obtain the theoretical impedance value from the measured one:

$$Z_{//} = \mathrm{Re}(Z_{\log})\frac{\ln(b/s)}{\ln(d/s)} + j\,\mathrm{Im}(Z_{\log}) \qquad (17)$$

The procedure presented in this paper can also be extended to the study of the transverse impedance and used for structures more complex than this simple example.

5 REFERENCES

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