

RESISTIVE-WALL COUPLED-BUNCH BEAM BREAKUP*

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Abstract

The beam breakup problem of a series of point-like bunches passing through a circular resistive wall pipe is treated in this paper. The solution to the problem is given as an integral representation. The asymptotic behavior of the solution after long time of operation is obtained.

1 INTRODUCTION

The insertion devices of the proposed PERL light source [1] consist of twelve wigglers, each twelve meters long, and totally 144 meters. The beam is shielded from the surrounding environment by circular copper pipes of radius $b = 2.5$ mm. These pipes are in turn placed inside of the wigglers. The proposed injection cycle is twelve hours. One may ask whether the beam can survive its interaction with the beam-induced resistive-wall wakefield for such a long time, particularly when the beam pipe radius b is so small. We address this problem in this paper.

The paper is organized as follows: In Section 2, the equation of motion for the beam breakup problem (bbu problem) is established, and the eigenvalue problem associated with this equation is solved. In Section 3, we solve the initial value problem; the solution consists of an integral representation of the transverse position of the M^{th} bunch at the longitudinal position z in terms of the eigensolution [2] and the initial bunch position of the preceding bunches. The asymptotic behavior, $M \rightarrow \infty$, of the integral representation is found in Section 4. In this paper, we treat only the case where only the leading bunch corresponding to $M = 0$ is misaligned initially [3]. In Section 5, we give a brief numerical discussion based on the proposed PERL parameters.

2 EIGENVALUE PROBLEM

An electron beam consisting of a series of identical point like bunches passes through a circularly cylindrical pipe of radius b and conductivity σ . The entrance to the pipe is located at $z = 0$, and the M^{th} bunch, $M = 0, 1, 2, \dots$, moves in the z direction according to $z = ct + Mc\tau_B$, where $\tau_B \equiv 1/f_B$ is the bunch separation in units of seconds. We use y to denote either the vertical or the horizontal coordinate of the beam particle. Inside of the pipe, the equation of

motion for a particle in the bunch M is

$$y_M''(z) + k_y^2 y_M(z) = a \sum_{N=0}^{M-1} y_N(z) / \sqrt{M-N}, \quad (1)$$

where the right hand side of this equation represents the effects of the beam induced force, and $a = 4I_{av}\delta_{skin}/\gamma I_{Alven}b^3$ with $I_{av} = eN_B/\tau_B$, $eN_B =$ bunch charge, $I_{Alven} = 4\pi\epsilon_0 mc^3/e \simeq 17000 \text{ Amp}$, $\gamma =$ the relativistic energy factor, and skin depth $\delta_{skin} = \sqrt{2/\mu_0\sigma\omega_B}$. The factor $1/\sqrt{M-N}$ is a consequence of the fact that the transverse wakefield for $\tau = t - z/c$ is proportional to $1/\sqrt{\tau}$ [4]. We ignore the effects of the wake force of a bunch on itself; as a consequence, we take $M-1$ instead of M as the upper limit of the sum in (1).

The right hand side of (1) is a convolution sum, therefore, it can be diagonalized by a Fourier transform. The definitions $F(\theta) = \sum_{M=1}^{\infty} e^{iM\theta}/\sqrt{M}$ and $\xi(\theta, z) = \sum_{M=0}^{\infty} y_M(z)e^{iM\theta}$, together with (1) lead to

$$y_M(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\theta e^{-iM\theta} \xi(\theta, z), \quad (2)$$

and

$$\xi''(\theta, z) + k_y^2 \xi(\theta, z) = aF(\theta)\xi(\theta, z). \quad (3)$$

The function $F(\theta)$ can be written as [5]

$$F(\theta) = \sqrt{\frac{i\pi}{\theta}} + \sum_{n=0}^{\infty} \zeta(1/2 - n) \frac{(i\theta)^n}{n!}, \quad (4)$$

$$\cong \sqrt{\frac{i\pi}{\theta}} - 1.460 - 0.208i\theta + O(\theta^2) \quad (5)$$

where $\zeta(x)$ is Riemann's ζ function.

We see $F(\theta)$ has a branch point at $\theta = 0$. This implies, through (3), that $\xi(\theta, z)$ also has a singularity in θ at the same position. It follows from causality that this singularity lies below the contour of (2) [3].

The equation(3) can be regarded as an eigenvalue equation with different coherent solutions distinguished by different values of θ . The eigenvalue for the mode θ is

$$k_c(\theta) = \sqrt{k_y^2 - aF(\theta)}, \quad (6)$$

and the corresponding eigenvectors are $\cos[k_c(\theta)z]$, and $\sin[k_c(\theta)z]$. We solve below the transient beam breakup problem by relating it to the above coherent solutions.

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3 INITIAL VALUE PROBLEM

We find in this section the transient solution to the equation of motion (1). In other words, we find an expression for $y_M(z)$ in terms of the initial conditions $y_{M0} \equiv y_M(z=0)$ and $y'_{M0} \equiv y'_M(z=0)$.

We first note that the transient solution of (3) is

$$\xi(\theta, z) = \xi_0(\theta) \cos[k_c(\theta)z] + \frac{\xi'_0(\theta)}{k_c(\theta)} \sin[k_c(\theta)z], \quad (7)$$

expressing $\xi(\theta, z)$ in terms of the initial conditions $\xi_0(\theta)$ and $\xi'_0(\theta)$. Applying (2) to both sides of this equation, we obtain

$$\begin{aligned} y_M(z) &= y_{M0} \cos k_y z + y'_{M0} \sin k_y z / k_y \\ &+ \frac{1}{2\pi} \sum_{N=0}^{M-1} y_{N0} \int_{-\pi}^{\pi} d\theta e^{-i(M-N)\theta} \cos k_c(\theta) z \\ &+ \frac{1}{2\pi} \sum_{N=0}^{M-1} y'_{N0} \int_{-\pi}^{\pi} d\theta e^{-i(M-N)\theta} \\ &\quad \times \sin k_c(\theta) z / k_c(\theta). \end{aligned} \quad (8)$$

This is the transient solution of the equation of motion (1).

4 ASYMPTOTIC BEHAVIOR

In this paper, we study the equation (8) corresponding to the case where only the first bunch corresponding to $M=0$ is initially excited. Specifically, we assume $y_{M0} = y_{00}\delta_{M,0}$, and $y'_{M0} = 0, \forall M$. Then for $M \neq 0$, the equation (8) becomes

$$y_M(z) = \frac{1}{2\pi} y_{00} \int_{-\pi}^{\pi} d\theta e^{-iM\theta} \cos k_c(\theta) z. \quad (9)$$

The following representation for the above equation is useful:

$$y_M(z) = y_{00} \eta_M(z) = y_{00} [\eta_M^{(+)}(z) + \eta_M^{(-)}(z)], \quad (10)$$

where

$$\eta_M^{\pm} = \frac{1}{4\pi} \int_{-\pi}^{\pi} d\theta e^{\Phi_M^{\pm}(\theta)}, \quad (11)$$

with

$$\Phi_M^{\pm}(\theta) = -iM\theta \pm ik_c(\theta)z. \quad (12)$$

We wish to find the asymptotic behavior of (9), or equivalently of (11), when $M \rightarrow \infty$. The method we use is that of saddle point. It is well known that the asymptotic behavior of the integral (9) is determined by the behavior of $\cos k_c(\theta)$ near $\theta=0$. In other words, the saddle point $\theta_{saddle} \rightarrow 0$ as $M \rightarrow \infty$. Recall that the function $F(\theta)$ is given for small θ by (5). We restrict our discussion to the cases where this function inside the integrand of (9) can be approximated by the singular part $\sqrt{i\pi/\theta}$ in the effective saddle-point integration region. Thus, we set

$$k_c(\theta) = \sqrt{k_y^2 - a\sqrt{i\pi/\theta}} \quad (13)$$

throughout the rest of the paper. For PERL, as represented by Table 1 in Section 5, $|\sqrt{i\pi/\theta}| \sim 10^{-6} \ll 1.460$ for θ lying within the effective Gaussian integration region of the saddle point method.

In carrying out the asymptotic analysis below, we shall distinguish two cases:

First Case: This is the case where either $k_y = 0$, or M is so large that in the effective region of the integrand, the absolute value of the $\sqrt{i\pi/\theta}$ term in (13) $\gg k_y^2$. As a consequence, we can approximate for this case, $k_c(\theta) \cong a'i(\theta)^{1/4}$ with $a' = \sqrt{a\sqrt{\pi}}$. This case will be referred to as the no focusing case. The following condition has to be satisfied in order for the non-focusing approximation to be applicable: $\chi_N \equiv |a\sqrt{i\pi/\theta_N}|/k_y^2 \gg 1$, where we use the notation θ_N for θ_{saddle} in this case.

Second Case: In this case, k_y^2 in (13) dominates over the $\sqrt{i\pi/\theta}$ term. Therefore, $k_c(\theta) \cong k_y - 2a''\sqrt{i/\theta}$, where $a'' = a\sqrt{\pi}/4k_y$. This case will be referred to as the strong focusing case. The condition for the validity of this approximation is $\chi_S \equiv k_y^2/|a\sqrt{i\pi/\theta_S}| \gg 1$, where we use the notation θ_S for θ_{saddle} in this case.

4.1 No Focusing (N) case

The relevant equations are (10) and (11) with

$$\Phi_M^{\pm}(\theta) = -iM\theta \mp a'z \left(\frac{i}{\theta}\right)^{1/4}. \quad (14)$$

In locating the saddle points, it suffices to discuss only the function $\Phi_M^{(+)}(\theta)$, since $\Phi_M^{(-)}(\theta)$ is related to $\Phi_M^{(+)}(\theta)$ by an analytic continuation. The condition $\Phi_M^{(+)}(\theta_N) = 0$ gives the following five saddle points:

$$\theta_N = -i|\theta_N|h_N, \quad (15)$$

where

$$h_N = (e^{i\pi/5}, e^{i3\pi/5}, -1, e^{-i\pi/5}, e^{-i3\pi/5}), \quad (16)$$

and

$$|\theta_N| = \left(\frac{a'z}{4M}\right)^{4/5}. \quad (17)$$

We need the second derivative of $\Phi_M^{(+)}$ evaluated at the saddle points. They are

$$\Phi_M^{(+)}(\theta_N) = \frac{5a'z}{16} \frac{1}{|\theta_N|^{9/4}} h_N^*, \quad (18)$$

where h_N^* is the complex conjugate of (16).

The most important contribution to (11) comes from the saddle point corresponding to the third element of (16). After the routine saddle-point Gaussian integration, followed by some change of variables, we obtain the following asymptotic result:

$$\eta(t) = \frac{2}{5\sqrt{2\pi}} \frac{\tau_B}{t_N} \left(\frac{t_N}{t}\right)^{9/10} \exp[(t/t_N)^{1/5}], \quad (19)$$

where we have set $t = M\tau_B$ and used the notation $\eta(t)$ for $\eta_M(z)$, and the growth time

$$t_N = \tau_B \frac{1}{4} \left(\frac{4}{5} \right)^5 \frac{1}{a'^4 z^4}. \quad (20)$$

4.2 Strong Focusing (S) case

Again, we treat (10) and (11), but

$$\Phi_M^{(\pm)}(\theta) = -iM\theta \pm iz \left(k_y - 2a'' \sqrt{i/\theta} \right). \quad (21)$$

The asymptotic treatment of this case is similar to that of the no-focusing case. The condition $\Phi_M^{(+)}(\theta_S) = 0$ gives the following three saddle points:

$$\theta_S = |\theta_S| (e^{i\pi/6}, e^{i5\pi/6}, -i), \quad (22)$$

where

$$|\theta_S| = \left(\frac{a''z}{M} \right)^{2/3}, \quad (23)$$

and the corresponding second derivatives are

$$\Phi_M^{(+)}(\theta_S) = \frac{3a''z}{2|\theta_S|^{5/2}} (e^{-i2\pi/3}, e^{+i2\pi/3}, 1). \quad (24)$$

The standard saddle point integration gives the leading asymptotic contribution

$$\begin{aligned} \eta(t) &= \frac{4}{3\sqrt{2\pi}} \frac{\tau_B}{t_S} \left(\frac{t_S}{t} \right)^{5/6} \exp[(t/t_S)^{1/3}] \\ &\times \cos[\sqrt{3}(t/t_S)^{1/3} + \pi/6] \cos[k_y z], \end{aligned} \quad (25)$$

where the growth time

$$t_S = \tau_B \frac{2^7}{3^3} \frac{k_y^2}{a'^4 z^2}. \quad (26)$$

That, from (20) and (26), $t_S \propto (k_y^2 z^2) t_N$ is interesting.

5 AN EXAMPLE: PERL

We start by examining the bbu effect on the PERL beam passing through the insertion devices [1]. A preliminary parameter list of PERL beam is given in Table 1. For simplicity, we assume that there are no gaps between the insertion devices; in other words, we approximate the twelve insertion devices with gaps in between by a continuous pipe of 144 meters.

There is no horizontal focusing provided by a planar wiggler. Consequently, no focusing approximation should be used for bbu problem in the horizontal plane. Numerical computation gives $\eta(t) \sim 10^{89}$ and the growth time $t_N \sim 60$ ns. Clearly, bbu can not be ignored for PERL. The situation can be improved by increasing the pipe radius b and by providing some external focusing k_y .

We try $k_y = 3$ and $b = 3$ mm; values of the other parameters of Table 1 remain unchanged. We obtain $\chi_N \sim 0.1$ and $\chi_S \sim 6$. The Strong-focusing approximation

Table 1: PERL parameters

f_B	1.3 GHz
b	2.5 mm
pipe length	144 m
conductivity σ	6×10^7 / (Ohm meter)
beam energy	3 GeV
bunch charge	.15 nC
$t = M\tau_B$	12 hours

is marginally applicable. Let's apply it. The results are $\eta(t) \sim 1$ and $t_S \sim 2$ seconds. The number $\eta(t) \sim 1$ is actually an overly optimistic estimate. In Obtaining (25), we assumed that only the leading bunch corresponding to $M = 0$ is initially dislocated. It can be shown [3] that had we assumed all the bunches to be initially misaligned, then the result would have been different; for a large M , the coefficient in front of the exponential factor in (25) would have been multiplied by a large number. Fortunately, (26) as well as the exponential and the sinusoidal factors in (25) are consequences of the eigenvalue problem; they are independent of the initial condition. In summary of this paragraph, $t_S \sim 2$ seconds is a valid estimate of the growth time corresponding to the beam parameters we have chosen, irrespective of the initial condition. With such a large growth time, it should be easy to take care of the bbu problem for PERL by designing an appropriate feedback damper.

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6 REFERENCES

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