A CONFORMAL SCHEME FOR WAKE FIELD CALCULATION

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Abstract

Existing computer codes experience severe problems in wake field calculations for long structures. To overcome these a new implicit scheme for rotationally symmetric geometry has been developed. Unlike previous conformal approaches this scheme has a second order convergence without the need to reduce the maximal stable time step of conventional staircase method. It features zero dispersion in longitudinal direction and by using a moving mesh it allows for wake potential calculation for very long structures. Several numerical examples are presented.

1 INTRODUCTION

The finite difference methods are applied successfully for calculation of wake fields in accelerators [1]. However the existing computer codes experience severe problems in short range wake field calculation for ultra short bunches [2]. Two main sources of the problems are the grid dispersion and the staircase geometry approximation.

To develop the scheme without dispersion in longitudinal direction we used the idea of the paper [3], namely to split the space operator in the transversal and the longitudinal parts and to use an implicit scheme based on the transversal part. The original staircase scheme of the work [3] allows only calculation for the fully rotationally symmetric case with bunches moving along the axis. As we show in this paper the same idea can be used for the higher order azimuthal modes as well. The new scheme is conditionally stable and allows to use the "magic" time step, equal to the space step in the longitudinal direction divided by the speed of light.

To overcome the staircase problem a conformal scheme described in the paper [4] is used. Unlike other conformal approaches this scheme is second order convergent without the need to reduce the maximal stable time step of the conventional staircase method. This feature allows to use a moving mesh and the "magic" time step in the new implicit conformal scheme.

2 WAVE EQUATION

We consider a perfectly conducting structure S and assume that the bunch is moving in domain Ω with the velocity of light $\mathbf{c}$ and is characterized by a charge distribution ρ. The bunch introduces an electric current $\mathbf{j}=\mathbf{c}\rho$ and thus we have to solve for

$$\nabla\times\mathbf{H}=\frac{\partial}{\partial t}\mathbf{D}+\mathbf{j},\ \nabla\times\mathbf{E}=-\frac{\partial}{\partial t}\mathbf{B},\nabla\cdot\mathbf{D}=\rho,\ \nabla\cdot\mathbf{B}=0 \quad (1)$$

$$\mathbf{H}=\mu^{-1}\mathbf{B},\mathbf{D}=\varepsilon\mathbf{E},\ x\in\Omega,\qquad \mathbf{n}\times\mathbf{E}=0,\ x\in S$$

The full field $\mathbf{D},\mathbf{H}$ can be decomposed into the field of the bunch in free space $\mathbf{D}^0,\mathbf{H}^0$ and a scattered field

$$\mathbf{D}^s=\mathbf{D}-\mathbf{D}^0,\qquad \mathbf{H}^s=\mathbf{H}-\mathbf{H}^0. \quad (2)$$

The scattered field can be presented by vector potential $\mathbf{A}$:

$$\mathbf{D}^s=\nabla\times\mathbf{A},\qquad \mathbf{H}^s=\frac{\partial}{\partial t}\mathbf{A}. \quad (3)$$

Substitution of the presentation (3) in the system (1) gives the problem for the vector potential $\mathbf{A}$

$$\nabla\times\nabla\times\mathbf{A}=-\frac{\partial^2}{c^2\partial t^2}\mathbf{A},\quad \nabla\cdot\mathbf{A}=0,\quad x\in\Omega, \quad (4)$$

$$\mathbf{n}\cdot\mathbf{A}=\int_{-\infty}^{t}\mathbf{n}\cdot\mathbf{H}^0 d\tau,\ \mathbf{n}\times\nabla\times\mathbf{A}=-\int_{-\infty}^{t}\mathbf{n}\times\nabla\times\mathbf{H}^0 d\tau,\ x\in S.$$

3 IMPLICIT SCHEME

The new scheme will be introduced in context of Finite Integration Technique [5]. Starting from Maxwell's equation in integral form and introducing decomposition of the computation domain into a collection of cells, we obtain a set of discrete equations on a grid doublet [5]:

$$\mathbf{C}\hat{\mathbf{e}}=-\frac{d}{dt}\hat{\mathbf{b}},\ \tilde{\mathbf{C}}\hat{\mathbf{h}}=\frac{d}{dt}\hat{\mathbf{d}}+\hat{\mathbf{j}},\quad \mathbf{S}\hat{\mathbf{b}}=\mathbf{0},\ \tilde{\mathbf{S}}\hat{\mathbf{d}}=\mathbf{q}. \quad (5)$$

They are completed by the discrete form of the material relations $\hat{\mathbf{e}}=\mathbf{M}_{\varepsilon^{-1}}\hat{\mathbf{d}}$, $\hat{\mathbf{h}}=\mathbf{M}_{\mu^{-1}}\hat{\mathbf{b}}$. To establish a time-stepping algorithm we can approximate the time derivatives in (5) by central difference expressions with $t_n=t_0+n\Delta t$:

$$\begin{aligned}&\hat{\mathbf{h}}^n=\hat{\mathbf{h}}^n-\Delta t\mathbf{M}_{\mu^{-1}}\mathbf{C}\hat{\mathbf{e}}^{n+1/2}, \quad (6)\\ &\hat{\mathbf{e}}^{n+1/2}=\hat{\mathbf{e}}^{n-1/2}+\Delta t\mathbf{M}_{\varepsilon^{-1}}(\mathbf{C}_1^T\bar{\hat{\mathbf{h}}}^n+\mathbf{C}_2^T\hat{\mathbf{h}}^n-\hat{\bar{\mathbf{j}}}^n),\\ &\bar{\hat{\mathbf{h}}}^n\equiv\theta\hat{\mathbf{h}}^{n+1}+(1-2\theta)\hat{\mathbf{h}}^n+\theta\hat{\mathbf{h}}^{n-1},\end{aligned}$$

where we have split the operator $\tilde{\mathbf{C}}=\mathbf{C}^T$ into the transversal operator $\mathbf{C}_1^T$ and the longitudinal operator $\mathbf{C}_2^T$ and θ is a numerical parameter to be defined. If we note the longitudinal coordinate by z and the transversal coordinates by r,φ, the operators have the form

$$\mathbf{C}_1^T=\begin{pmatrix}\mathbf{0}&\mathbf{0}&-\mathbf{P}_\varphi^T\\ \mathbf{0}&\mathbf{0}&\mathbf{P}_r^T\\ \mathbf{P}_\varphi^T&-\mathbf{P}_r^T&\mathbf{0}\end{pmatrix},\ \mathbf{C}_2^T=\begin{pmatrix}\mathbf{0}&\mathbf{P}_z^T&\mathbf{0}\\ -\mathbf{P}_z^T&\mathbf{0}&\mathbf{0}\\ \mathbf{0}&\mathbf{0}&\mathbf{0}\end{pmatrix}.$$

From the system (6) we receive a numerical scheme for the vector potential $\mathbf{A}$:

$$\begin{aligned}&(\mathbf{I}+\theta\mathbf{T})\mathbf{a}^{n+1}=2\mathbf{a}^n-\mathbf{a}^{n-1}-\\ &\quad-\mathbf{T}\left((1-2\theta)\mathbf{a}^n+\theta\mathbf{a}^{n-1}\right)-\mathbf{L}\mathbf{a}^n+\mathbf{F}^n, \quad (7)\\ &\mathbf{T}=\Delta t^2\mathbf{M}_{\mu^{-1}}\mathbf{C}\mathbf{M}_{\varepsilon^{-1}}\mathbf{C}_1^T,\ \mathbf{L}=\Delta t^2\mathbf{M}_{\mu^{-1}}\mathbf{C}\mathbf{M}_{\varepsilon^{-1}}\mathbf{C}_2^T,\\ &\mathbf{F}^n=-(\mathbf{I}+\theta\mathbf{T})\mathbf{a}_0^{n+1}+2\mathbf{a}_0^n-\mathbf{a}_0^{n-1}-\\ &\quad-\mathbf{T}\left((1-2\theta)\mathbf{a}_0^n+\theta\mathbf{a}_0^{n-1}\right)-\mathbf{L}\mathbf{a}_0^n,\end{aligned}$$

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$$\mathbf{a}^n = \int_{-\infty}^{t_n} \widehat{\mathbf{h}}_s d\tau\,, \qquad \mathbf{a}_0^n = \int_{-\infty}^{t_n} \widehat{\mathbf{h}}_0 d\tau\,,$$

where the vectors $\widehat{\mathbf{h}}_0$, $\widehat{\mathbf{h}}_s$ correspond to the fields $\mathbf{H}^0$, $\mathbf{H}^s$ in the representation (2). This scheme approximates the problem (4). $\mathbf{F}^{(m)}$ approximates the boundary conditions.

It is easy to show [4] that all eigensolutions of the spatial discretization correspond to non-dissipative and non-growing oscillations with a real-valued circular frequency $\omega_j = \sqrt{\lambda_j}$ and the time-dependency $\widehat{\mathbf{h}}(t) \propto \mathrm{Re}\left\{e^{i\omega_j t}\right\}$.

The next step in the stability analysis is the stability of the time-stepping scheme. A spectral stability condition is

$$c\Delta t \le \Delta z, \quad 0.25 \le \theta. \tag{8}$$

With the time step $c\Delta t = \Delta z$ allowed by condition (8) the scheme has no dispersion in the longitudinal direction and a moving mesh can be employed easily. The results in this case are fully equivalent to a stationary mesh as no interpolation is necessary.

To reduce dispersion in the transversal direction we should use minimal value of θ. In our numerical examples we used $\theta = 1/3$.

4 GEOMETRY OF REVOLUTION

In this section we describe the realization of the above scheme for the case of a rotationally symmetric geometry.

For a bunch moving offset a from and at speed of light c parallel to the axis of a rotationally symmetric structure, the source current $\mathbf{j}$ can be presented as

$$\mathbf{j} = \frac{c\lambda(z/c - t)\delta(r-a)}{\pi a} \sum_{m=0}^{\infty} \frac{\cos m\varphi}{1+\delta_{m0}} \mathbf{z}$$

where $\lambda(s)$ is the longitudinal charge distribution.

The numerical scheme (7) for mode m has the form

$$\begin{aligned}
&\mathbf{M}_{\mu_r^{-1}}^{-1}\Delta t^{-2}\left(\mathbf{a}_r^{n+1} - 2\mathbf{a}_r^n + \mathbf{a}_r^{n-1}\right) = \mathbf{P}_z\mathbf{M}_{\varepsilon_\varphi^{-1}}\mathbf{P}_z^T\mathbf{a}_r^n + \mathbf{F}_r^n + \\
&\quad + \left(-m^2\mathbf{M}_{\varepsilon_z^{-1}}\overline{\mathbf{a}}_r^n - \mathbf{P}_z\mathbf{M}_{\varepsilon_\varphi^{-1}}\mathbf{P}_r^T\overline{\mathbf{a}}_z^n + m\mathbf{M}_{\varepsilon_z^{-1}}\mathbf{P}_r^T\overline{\mathbf{a}}_\varphi^n\right), \\
&\mathbf{M}_{\mu_\varphi^{-1}}^{-1}\Delta t^{-2}\left(\mathbf{a}_\varphi^{n+1} - 2\mathbf{a}_\varphi^n + \mathbf{a}_\varphi^{n-1}\right) = \mathbf{P}_z\mathbf{M}_{\varepsilon_r^{-1}}\mathbf{P}_z^T\mathbf{a}_\varphi^n + \mathbf{F}_\varphi^n + \\
&\quad + \left(\mathbf{P}_r\mathbf{M}_{\varepsilon_z^{-1}}\mathbf{P}_r^T\overline{\mathbf{a}}_\varphi^n - m\mathbf{P}_z\mathbf{M}_{\varepsilon_r^{-1}}\overline{\mathbf{a}}_z^n - m\mathbf{P}_r\mathbf{M}_{\varepsilon_z^{-1}}\overline{\mathbf{a}}_r^n\right), \\
&\mathbf{M}_{\mu_z^{-1}}^{-1}\Delta t^{-2}\left(\mathbf{a}_z^{n+1} - 2\mathbf{a}_z^n + \mathbf{a}_z^{n-1}\right) = m\mathbf{M}_{\varepsilon_r^{-1}}\mathbf{P}_z^T\mathbf{a}_\varphi^n - \\
&-\mathbf{P}_r\mathbf{M}_{\varepsilon_\varphi^{-1}}\mathbf{P}_z^T\mathbf{a}_r^n + \mathbf{F}_z^n + \left(\mathbf{P}_r\mathbf{M}_{\varepsilon_\varphi^{-1}}\mathbf{P}_r^T - m^2\mathbf{M}_{\varepsilon_r^{-1}}\right)\overline{\mathbf{a}}_z^n,
\end{aligned} \tag{9}$$

where $\overline{\mathbf{a}}_p^n \equiv \theta\mathbf{a}_p^{n+1} + (1-2\theta)\mathbf{a}_p^n + \theta\mathbf{a}_p^{n-1}$, $p = r, \varphi, z$. Note that $\mathbf{F}_z^n \equiv 0$ if the bunch moves parallel to the axis.

When a bunch moves along the axis, only the $\mathbf{A}_\varphi$ component is different from zero and our scheme with $\theta = 0.5$ is reduced to the scheme of the paper [3].

In the general case, at the first step we calculate the vector $\mathbf{a}_z^{n+1}$ and have to solve the linear system with the matrix $\mathbf{I} - \Delta t^2\theta\mathbf{M}_{\mu_z^{-1}}\mathbf{P}_r\mathbf{M}_{e_j^{-1}}\mathbf{P}_r^T + m^2\Delta t^2\theta\mathbf{M}_{\mu_z^{-1}}\mathbf{M}_{\varepsilon_r^{-1}}$. This matrix is a block diagonal one with N_z blocks. Each block is a three band matrix of size N_r and can be resolved by $O(N_r)$ operation.

Now we can use the component $\mathbf{a}_z^{n+1}$ in the equations for components $\mathbf{a}_r^{n+1}$, $\mathbf{a}_\varphi^{n+1}$ which are coupled. We have to solve a system with the matrix

$$\begin{pmatrix} \mathbf{I} + \Delta t^2\theta m^2\mathbf{M}_{\mu_r^{-1}}\mathbf{M}_{e_z^{-1}} & m\theta\mathbf{M}_{\mu_r^{-1}}\mathbf{M}_{\varepsilon_z^{-1}}\mathbf{P}_r^T \\ m\theta\mathbf{M}_{\mu_\varphi^{-1}}\mathbf{P}_r\mathbf{M}_{\varepsilon_z^{-1}} & \mathbf{I} - \Delta t^2\theta\mathbf{M}_{\mu_\varphi^{-1}}\mathbf{P}_r\mathbf{M}_{e_z^{-1}}\mathbf{P}_r^T \end{pmatrix},$$

which is a block diagonal one with N_z blocks. Each block is of size $2N_r$ and has a seven band structure as shown in Fig.1.

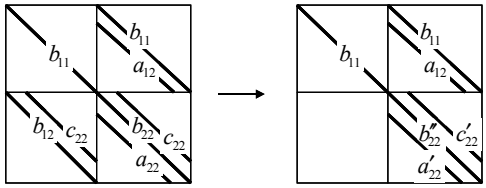

Fig. 1. Reduction of the matrix

It can be reduced to a three diagonal type by $O(N_r)$ operation. This means its resolving takes only $O(N_r N_z)$ operations and the algorithm demands the same order of operations as the explicit method.

Note that because of material matrices in staircase approximation are diagonal it is possible to use the divergence relation $\mathbf{a}_\varphi = m^{-1}\mathbf{M}_{\mu_\varphi^{-1}}\left(\mathbf{P}_r\mathbf{M}_{\mu_r^{-1}}^{-1}\mathbf{a}_r + \mathbf{P}_z\mathbf{M}_{\mu_z^{-1}}^{-1}\mathbf{a}_z\right)$ and eliminate the $\mathbf{a}_\varphi$ component from the equation for the $\mathbf{a}_r$ component. This gives a more effective algorithm.

However the last relation is useless in the case of non-diagonal material matrices as used in the conformal scheme.

5 CONFORMAL SCHEME

With the standard staircase approximation of curved boundaries we receive only a first order convergent scheme in the L_2^h grid norm. To obtain a second order convergent scheme we use the approach of the paper [4].

We allow the cells of the grid to be only partially filled by a PEC material with an arbitrarily shaped interface. Since the area of the cells near the boundary is reduced, the time step in the conformal scheme has to be reduced, too. To overcome this problem and to receive a stable algorithm without reducing the time step we modify only the material matrix $\mathbf{M}_{\mu^{-1}}$ which is a composition of diagonal matrices $\mathbf{M}_{\mu^{-1}} = \tilde{\mathbf{R}}\mathbf{M}$, $\mathbf{M} = \left\|\mu_{pijk}^{-1}\right\|$, $\tilde{\mathbf{R}} = \left\|\tilde{r}_{pijk}^{-1}\right\|$, $\mu_{pijk} = \mu\dfrac{s_{pijk}}{S_{pijk}}$, $\tilde{r}_{pijk} = \dfrac{S_{pijk}}{\tilde{L}_{pijk}}$, where s denotes a reduced cell area [4]. A new material matrix $\tilde{\mathbf{M}}_{\mu^{-1}}$ is composed by the relation $\tilde{\mathbf{M}}_{\mu^{-1}} = \mathbf{V}^T\mathbf{D}\mathbf{V}$, where $\mathbf{D} = \tilde{\mathbf{R}}\mathbf{U} > 0$ is a di-

agonal matrix, responsible for the order of the approximation, and $\mathbf{V}$ is a matrix of weights. In [4] we described building of the matrix $\tilde{\mathbf{M}}_{\mu^{-1}}$ for explicit algorithm. Since the scheme is implicit in transversal direction we use weights only in longitudinal direction and only for facets in rz and φz planes.

The material matrix $\tilde{\mathbf{M}}_{\mu^{-1}}$ is a non-diagonal one. To be able to use the algorithm of section 4, we should modify the scheme (7). If we note by $\tilde{\mathbf{M}}^0_{\mu^{-1}}$ the diagonal part of the matrix $\tilde{\mathbf{M}}_{\mu^{-1}}$, the modification of the scheme (7) is

$$\begin{aligned}\left(\mathbf{I}+\theta\mathbf{T}^0\right)\mathbf{a}^{n+1} &= -\mathbf{T}^0\left((1-2\theta)\mathbf{a}^n+\theta\mathbf{a}^{n-1}\right)+ \\ &+2\mathbf{a}^n-\mathbf{a}^{n-1}-(\mathbf{T}^1+\mathbf{L})\mathbf{a}^n+\mathbf{F}^n,\end{aligned} \qquad (10)$$

$$\mathbf{T}^0=\Delta t^2\tilde{\mathbf{M}}^0_{\mu^{-1}}\mathbf{C}\mathbf{M}_{\varepsilon^{-1}}\mathbf{C}_1^T,\ \mathbf{T}^1=\Delta t^2\left(\tilde{\mathbf{M}}_{\mu^{-1}}-\tilde{\mathbf{M}}^0_{\mu^{-1}}\right)\mathbf{C}\mathbf{M}_{\varepsilon^{-1}}\mathbf{C}_1^T,$$

where the operator $\mathbf{T}^1$ is different from zero only in some boundary cells.

In all examples shown in the next section we used the scheme (10) with moving mesh.

6 NUMERICAL EXAMPLES

The conformal scheme (10) is included in a newly developed code. In the current version only the fully rotationally symmetric case ($m=0$) is realized. However, the higher order modes algorithm was tested using Matlab 6.0.

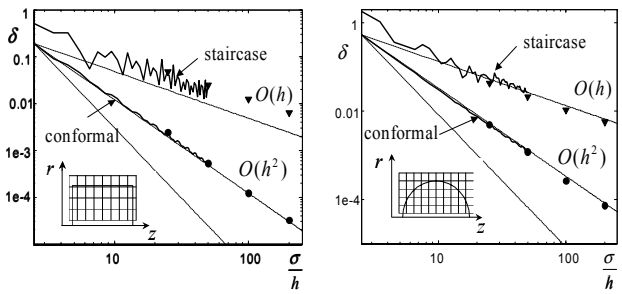

Fig.2. Error in loss factor for pillbox and sphere

To calculate the wake potential W_λ we used indirect method [6]. For $m=0$ the used formula is

$$W_\lambda(s)=-\frac{1}{Q}\int_C\left(\frac{1}{r}\frac{\partial}{\partial r}rA_\varphi dz+\left(\frac{\partial}{\partial s}A_\varphi-\frac{\partial}{\partial z}A_\varphi\right)dr\right)\Bigg|_{r,z,t=(z+s)/c}$$

Fig.2 shows the loss factor error $\delta=\left|L_{calc}-L\right|L^{-1}$ for a Gaussian bunch with $\sigma=0.5$ cm passing through a pillbox (Fig.2 left) and a spherical resonator (Fig.2 right). The pillbox has the length 1.8 cm and radius 0.9 cm. The sphere has the diameter 1.8 cm. The analytical loss factor L is equal to 0.589459 V/pC for the pillbox and 0.152446 V/pC for the sphere. The error for stationary mesh is demonstrated by lines. The results for moving mesh are shown by triangles and circles. In Fig. 3 (left) the geometry of a taper is drawn. In Fig. 3(right) the error δ relative to the extrapolated loss factor $L=$-7.63777V/pC for bunch with $\sigma=0.1$ cm is shown.

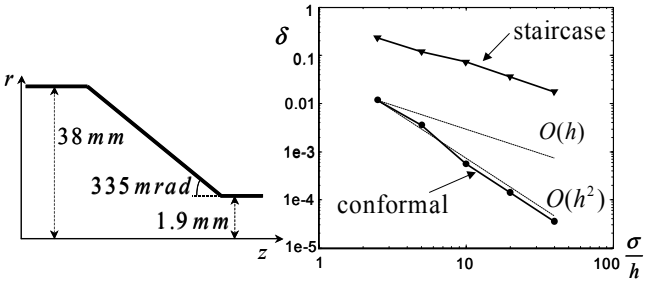

Fig.3. Error in loss factor for a taper

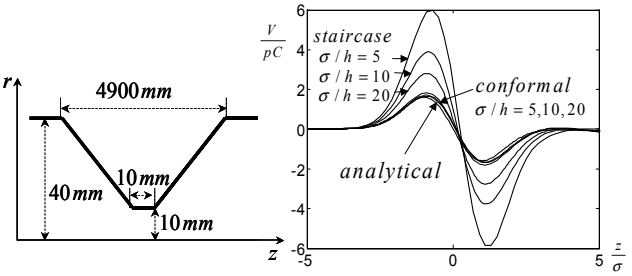

Fig.4.Wake potential of a collimator

Fig.4 demonstrates the wake potential for a collimator of 490 cm length and the bunch with $\sigma=0.1$ cm. The solution is compared to the analytical estimation [7].

The conformal scheme shows second order convergence and gives results of high accuracy with only 5 mesh steps per σ in all tests. Note that the staircase scheme in the last example gives at the same resolution an error in excess of 300 %.

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