CANCELLATION EFFECTS IN CSR INDUCED BUNCH TRANSVERSE DYNAMICS IN BENDS

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1 INTRODUCTION

The existence of the centrifugal space charge force F^{CSCF} was first pointed out by Talman [1] when studying the curvature induced transverse force on a coasting beam in a storage ring. The logarithmic dependence of this force on particle transverse offset could cause shifts in horizontal tune and contribute significantly to chromaticity for a coasting beam. These effects of logarithmic divergence in F^{CSCF} on transverse dynamics were later pointed out by Lee [2] to be cancelled by the effect of beam induced electric potential, which enters into the transverse dynamics through dispersion by changing the kinetic energy of the particles. As the result of the cancellation, the residual effect on a coasting beam is about σ_x/R times the F^{CSCF} effect, where σ_x is the transverse bunch size, and R is the equilibrium radius of the ring.

Even though the cancellation effect was cleared for *coasting beams*, it was a dispute again for the CSR induced transverse effect for *bunched beams*. In Ref. [3], it was concluded that for bunched beams, the effect of F^{CSCF} is no longer cancelled by the potential energy, and there exists a longitudinal force named non-inertial space charge force F^{NSCF} in addition to the usual longitudinal space charge and CSR forces. On the other hand, Derbenev pointed out [4] that for bunched beams, there is always the cancellation between the effect of F^{CSCF} on the transverse bunch dynamics and the effect of potential energy. Further analysis [5] shows that the accumulated effect of F^{NSCF} contributes to the potential energy; thus its effect on the transverse dynamics nearly cancels with that of F^{CSCF}. Most recently, the generality of the cancellation effect was questioned [6] and what exactly the cancellation is meant was under dispute again.

In this paper, we seek to clarify the meaning of the "cancellation effect" and its general application. By analyzing the generalized momentum and its dynamics, we show that the "centrifugal space charge force" arises as a result of the dependence of the metric on coordinates; in this sense it shares similar geometric feature as that of the usual centrifugal force. It turns out that for a charged particle in a bunch on a circular orbit, the usual centrifugal force—which is related to the *kinetic momentum*— always works together with the "centrifugal space charge force", and jointly they form the *generalized* centrifugal force—which is related to the *generalized momentum*. We show that in this *generalized* centrifugal force, the effect of the "centrifugal space charge force" is *always* cancelled by the potential energy effect; as a result, the effective terms after cancellation is free from logarithmic singularities caused by the nearby particle interaction. For both steady state and transient regimes, this cancellation is demonstrated using numerical simulation, and the behaviors of effective terms are presented.

2 PARTICLE DYNAMICS IN A ROTATING FRAME

Let us start with the Lagrangian of a particle in an electron bunch experiencing external and self-interaction fields, and then derive the dynamical equation of generalized momentum for particles moving on a circular orbit. In this way we illustrate how the "centrifugal space charge force" enters into the picture.

First, for a Cartesian coordinate system with 4-vectors $q = (ct, \mathbf{r})$, $U = dq/d\tau = (\gamma c, \gamma \mathbf{u})$ and 4-potential $A = (\Phi, \mathbf{A})$, the covariant form of the Lagrangian with Minkowski spacetime metric tensor $g_{\mu\nu}$ is

$$\mathcal{L} = -mc\sqrt{g_{\mu\nu}U^\mu U^\nu} + \mathcal{L}_{\text{int}} \tag{1}$$

with the interaction Lagrangian

$$\mathcal{L}_{\text{int}} = -\frac{e}{c}\, g_{\mu\nu}U^\mu A^\nu = -e\gamma(\Phi - \boldsymbol{\beta}\cdot\mathbf{A}). \tag{2}$$

The Euler-Lagrangian equation is

$$\frac{d}{d\tau}\frac{\partial \mathcal{L}}{\partial U^\mu} - \frac{\partial \mathcal{L}}{\partial q^\mu} = 0, \tag{3}$$

where the generalized momentums $P^\mu = -g^{\mu\nu}\partial\mathcal{L}/\partial U^\nu$ are

$$P^0 = (\gamma mc^2 + e\Phi)/c, \qquad \mathbf{P} = \gamma m\mathbf{u} + e\mathbf{A}/c. \tag{4}$$

Let $dt = \gamma d\tau$; we obtain the 3-dimensional projection of Eq. (3)

$$\frac{d\mathbf{P}}{dt} = -e(\nabla\Phi - \beta_i\nabla\cdot A_i), \tag{5}$$

and energy relation from the zeroth component of Eq. (3)

$$\frac{d(\gamma mc^2 + e\Phi)}{dt} = e\left(\frac{\partial\Phi}{\partial t} - \boldsymbol{\beta}\cdot\frac{\partial\mathbf{A}}{\partial t}\right). \tag{6}$$

The above discussion is based on a Cartesian coordinate system. Next we consider a bunch moving on a circular orbit. The particle dynamics in the bending plane is then expressed in terms of the cylindrical coordinates with respect to the center of the designed circular orbit: $\mathbf{r} = r\mathbf{e}_r + r\theta\mathbf{e}_s, \mathbf{u} = \dot{r}\mathbf{e}_r + r\dot{\theta}\mathbf{e}_s$. The relativistic Lagrangian in terms of cylindrical coordinates is then

$$\mathcal{L} = -mc^2\sqrt{1 - \frac{r^2\dot{\theta}^2 + \dot{r}^2}{c^2}} - e(\Phi - \frac{\dot{r}}{c}A_r - \frac{r\dot{\theta}}{c}A_s). \tag{7}$$

To compare with Eq. (5), we rewrite the Euler-Lagrangian equation from Eq. (7) in the following form:

$$\frac{dP_r}{dt} - v_s\frac{P_s}{r} = -e\left(\frac{\partial\Phi}{\partial r} - \boldsymbol{\beta}\cdot\frac{\partial\mathbf{A}}{\partial r}\right), \quad (8)$$

$$\frac{dP_s}{dt} + v_r\frac{P_s}{r} = -e\left(\frac{\partial\Phi}{r\partial\theta} - \boldsymbol{\beta}\cdot\frac{\partial\mathbf{A}}{r\partial\theta}\right), \quad (9)$$

where the generalized momentums P_r and P_s are defined as

$$P_r = p_r + eA_r/c, \qquad P_s = p_s + eA_s/c \quad (10)$$

with the kinetic momentums $p_r = \gamma m\dot{r}$ and $p_s = \gamma mr\dot{\theta}$. Comparing to Eq. (5), the right-hand sides in Eqs. (8) and (9) are the projection of the driving term in Eq. (5) to the basis of the rotating frame $(\mathbf{e}_r, \mathbf{e}_s)$, while the second terms on the left-hand sides in Eqs. (8) and (9) are purely due to the dependence of the metric on coordinates in the rotating coordinate system. Note that Eqs. (8) and (9) are readily reduced to Eq. (5) in straight sections where $r \to \infty$.

For a bunch with design energy $E_0 = \gamma_0 mc^2$ circulating on an orbit with design radius R, one has

$$\mathbf{B}^{\text{ext}} = \frac{mc^2}{e}\frac{\gamma_0\beta_0}{R}(\mathbf{e}_s\times\mathbf{e}_r), \quad \frac{eA_s^{\text{ext}}}{c} = -\gamma_0 m\beta_0 c\frac{r}{2R},$$

where $\beta_0 = (1-\gamma_0^{-2})^{1/2}$. With $A_s = A_s^{\text{ext}} + A_s^{\text{self}}$, the particle's transverse dynamics can be obtained from Eq. (8)

$$\frac{d\gamma m\dot{r}}{dt} = \beta_s c\left(\frac{\tilde{P}_s}{r} - \frac{\beta_0\gamma_0 mc}{R}\right) + F_r^{\text{eff}}. \quad (11)$$

Using Φ and $\mathbf{A}$ to represent only *self-interaction* potentials from now on, the part of the generalized momentum relating only to bunch self-interaction in Eq. (11) is

$$\tilde{P}_s = \gamma mc\beta_s + eA_s/c, \quad (12)$$

and the effective radial force F_r^{eff} in Eq. (11) is

$$F_r^{\text{eff}} = -e\left(\frac{\partial\Phi}{\partial r} - \boldsymbol{\beta}\cdot\frac{\partial\mathbf{A}}{\partial r}\right) - e\frac{dA_r}{cdt}. \quad (13)$$

Here we define $\beta_s c\tilde{P}_s/r$ in Eq. (11) as the "general centrifugal force"

$$F^{\text{GCF}} \equiv \beta_s c\frac{\tilde{P}_s}{r} = \gamma mr\dot{\theta}^2 + F^{\text{CSCF}}, \quad (14)$$

where $\gamma mr\dot{\theta}^2$ is the usual centrifugal force, and F^{CSCF} is the "centrifugal space charge force" due to the particles' collective interaction

$$F^{\text{CSCF}} = e\beta_s A_s/r. \quad (15)$$

Note here F^{CSCF} is centrifugal in direction, and it is a result of the metric dependence on coordinate due to the noninertial rotating frame—similar to the nature of the usual centrifugal force. Using $\mathbf{E} = -\nabla\Phi - \partial\mathbf{A}/c\partial t$ and $\mathbf{B} = \nabla\times\mathbf{A}$, one can show that F^{CSCF} and F_r^{eff} together give the total radial component of the Lorentz force

$$F_r^{\text{tot}} = (\mathbf{E} + \boldsymbol{\beta}\times\mathbf{B})\cdot\mathbf{e}_r = F^{\text{CSCF}} + F_r^{\text{eff}}. \quad (16)$$

Even though here F_r^{tot} is dominated by F^{CSCF} and thus is centrifugal in direction, F^{CSCF} is singled out as the "centrifugal space charge force" due to its geometrical nature.

Next we show that in $\tilde{P}_s$ of Eq. (12), the term eA_s/c—which represents the effect of F^{CSCF}— always works counteractively with the potential energy effect. With the definition of effective parallel force (parallel to $\beta_r\mathbf{e}_r + \beta_s\mathbf{e}_s$)

$$F_\parallel^{\text{eff}} = e\left(\frac{\partial\Phi}{c\partial t} - \boldsymbol{\beta}\cdot\frac{\partial\mathbf{A}}{c\partial t}\right), \quad (17)$$

Eq. (6) becomes

$$\frac{d\gamma mc^2}{cdt} = \boldsymbol{\beta}\cdot\mathbf{F} = -e\frac{d\Phi}{cdt} + F_\parallel^{\text{eff}}, \quad (18)$$

which can be integrated as

$$\gamma mc^2 = \gamma_0 mc^2 + \Delta E^{\text{tot}}(t_0) + \int_{t_0}^{t} F_\parallel^{\text{eff}}(t')cdt' - e\Phi(t), \quad (19)$$

with $\Delta E^{\text{tot}}(t_0)$ the initial kinetic and potential energy deviation from design energy

$$\Delta E^{\text{tot}}(t_0) = [\gamma(t_0)mc^2 - \gamma_0 mc^2] + e\Phi(t_0). \quad (20)$$

As a result, we have by combining Eq. (19) with Eq. (12)

$$\tilde{P}_s = \beta_s\gamma_0 mc + \beta_s\Delta E^{\text{tot}}(t_0)/c + \frac{\beta_s}{c}\int_{t_0}^{t} F_\parallel^{\text{eff}}(t')cdt' + \frac{e}{c}(A_s - \beta_s\Phi), \quad (21)$$

with $\Delta E^{\text{tot}}(t_0)$ given in Eq. (20). Applying Eq. (21) to Eq. (11), one gets the equation of motion which contains clearly the $(A_s - \beta_s\Phi)$ term:

$$\frac{d\gamma m\dot{r}}{dt} - \beta_s c\left(\frac{\beta_s\gamma_0 mc}{r} - \frac{\beta_0\gamma_0 mc}{R}\right) = G_0 + G_c + G_\parallel + G_r \quad (22)$$

with $$G_0 = \beta_s^2\frac{\Delta E^{\text{tot}}(t_0)}{r}, \quad G_\parallel = \frac{\beta_s^2}{r}\int_{t_0}^{t} F_\parallel^{\text{eff}}(t')cdt',$$

$$G_c = e\beta_s\frac{A_s - \beta_s\Phi}{r} = F^{\text{CSCF}} - e\beta_s^2\frac{\Phi}{r}, \quad G_r = F_r^{\text{eff}}$$

Note that Eq. (22) does not contain any approximation, which shows that the transverse dynamics of an electron is driven by the initial *total* energy deviation from design energy (G_0 term), the *effective* forces ($G_\parallel$ and G_r terms), and the residual of $(A_s - \Phi)$ (G_c term). It should be emphasized that the "cancellation effect" means A_s and Φ in G_c is nearly cancelled, where the A_s term represents the effect of F^{CSCF}. Typically, A_s and Φ in G_c have logarithmic dependence on the particle's transverse offset due to local (immediate neighbor) interaction. However, the residual term after their cancellation, G_c, is free from the logarithmic singularity. Our simulation in the next section shows that G_c is *always* negligible compared to $G_\parallel$ and G_r .

It is interesting to note that for the transient regime of a line bunch entering a circle, $-d\Phi/dt$ does not exhibit logarithmic behavior [7, 8]; neither does $\Phi(t) - \Phi(t_0)$. In this case, apart from the γ^{-2} dependence, $\Phi(t_0)$, $\Phi(t)$ and $A_s(t)$ have the same logarithmic behavior; therefore their differences are free from the logarithmic singularity. The

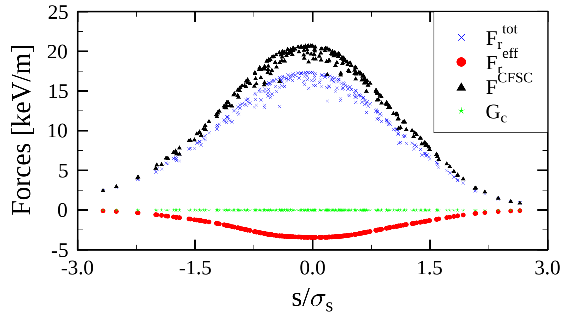

Figure 1: Force terms in Eq. (16) and G_c vs. longitudinal position for a Gaussian bunch.

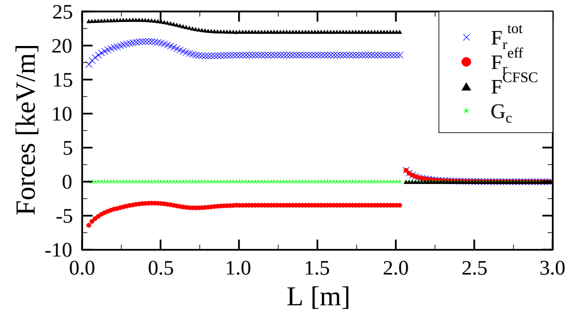

Figure 2: Force terms in Eq. (16) and G_c vs. pathlength. Note the entrance behavior at $L < 1$ m, and the exit behavior at $L > 2$ m.

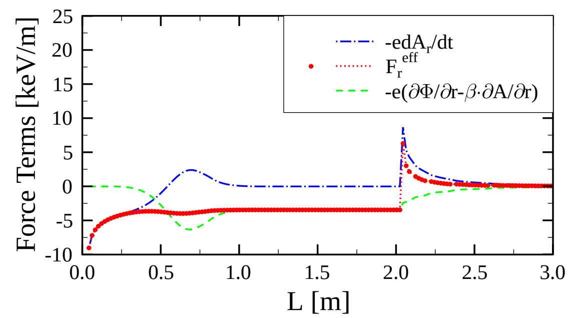

Figure 3: Terms contribute to $F_r^{\rm eff}$ in Eq. (13).

initial potential $\Phi(t_0)$ enters into G_0 of Eq. (22), which acts as the initial energy spread, and does not directly cause emittance growth for an achromatic bending system. However, for a non-achromatic bending system such as a single bend in a spectrometer, $\Phi(t_0)$ in G_0 can cause some observable effects on particles' transverse position, and one should be careful with data analysis in such cases. In general, when the bunch is not rigid in bends (such as in Fig. 1), $\Phi(t)$ and $\Phi(t_0)$ no longer have the same logarithmic behavior, while it can be shown analytically that $A_s(t)$ and $\Phi(t)$ are always the same in logarithmic dependence. Therefore the cancellation of logarithmic dependences of $A_s(t)$ and $\Phi(t)$ in G_c always holds.

3 SIMULATION RESULTS

The cancellation effect and behavior of residual terms in a steady state case have been analyzed earlier [5]. Here we use simulation [5] to show how it works in *both* steady state and transient regimes, including a bunch entering a circle from a straight path or exiting a circle to a straight path. For this purpose, we let a 5 GeV electron bunch with Gaussian longitudinal distribution and rms bunch length 0.2 mm move from a straight path to a bend of 10 m radius. The bunch charge is 1 nC. After $L = 2$ m ($11.5\,$deg) of bending, the bunch exits onto a straight path again. The numerical results of various force terms across the bunch at L=1.6 m are displayed in Fig. 1, where the bunch transverse size is 72 μm. The spread of $F_r^{\rm tot}$ and $F^{\rm CSCF}$ in Fig. 1 for fixed s/σ_s is due to their rapid dependence on transverse offset originated from the logarithmic behavior, which was a big concern for coasting beams [1] and later was proved to be not effective [2]. It is shown in Fig. 1 that for a bunched beam the effective centripetal radial force $F_r^{\rm eff}$ has negligible dependence on transverse offset. The G_c term has magnitude of 4×10^{-3} keV/m, so it appears to be zero in Fig. 1. In Fig. 2, we track the forces following a single particle in the bunch to show the transient behaviors of the force terms. Here in order to show a clean steady state result (so the particle potential does not change due to its internal motion), we choose a rigid Gaussian line bunch, and the particle dynamics does not respond to CSR force. Here again G_c is practically zero through all the transient and steady state regimes, indicating cancellation of the effect of $F^{\rm CSCF}$ with the potential energy effect. In Fig. 3, we show that in Eq. (13), $-edA_r/cdt$ is almost discontinuous at entrance and exit of a circle, and it remains zero in steady state, while $-e(\partial\Phi/cdt - \boldsymbol{\beta}\cdot\partial\mathbf{A}/cdt)$ is continuous throughout the transient and steady state regions. It should be emphasized that even though the logarithmic dependences are cleanly cancelled out and thus G_c is negligible, G_r or $F_r^{\rm eff}$ may have non-negligible effect on bunch transverse dynamics.

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