ON THE MAINTENANCE CONDITION OF A CRYSTALLINE BEAM

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Abstract

It has been often stated that the so-called "maintenance condition" must be satisfied to achieve beam crystallization in a cooler storage ring. The condition requires the ring lattice to have a betatron phase advance below about 127 degrees per single superperiod [1]. In the present work, we show that this condition is not sufficient in general situations. Systematic multi-particle simulations and analytic studies suggest that the phase advance per lattice period should be lower than 90 degrees in order to reach a crystalline state at an arbitrary line density.

1 INTRODUCTION

Since an anomaly in an electron-cooled proton beam was observed by the Novosibirsk group over twenty years ago [2], continuous discussion has been made as to whether one can really crystallize a charged-particle beam in a storage ring. In the middle 1980's, Schiffer and co-workers employed the molecular dynamics (MD) approach for a systematic study of Coulomb ordering [3]. Using a harmonic potential model, they first demonstrated the possibility of beam crystallization. Their excellent work was generalized later by Wei, Li, and Sessler who incorporated the characteristics of actual storage rings like alternating gradient focusing into MD calculations [4]. It is now widely believed that beam crystallization is, in principle, achievable if one can provide a strong three-dimensional (3D) cooling force.

Ordinary ion beams in accelerators are initially quite hot and thin in all three degrees of freedom. The effective betatron tune of each individual particle is close to its design value ν_0 uniquely determined by the machine lattice. However, once a cooling process is initiated, the effective tune is gradually reduced because the growing Coulomb interactions among stored particles hinder the free betatron oscillations. As a result, the beam may encounter resonance stopbands before arriving at the equilibrium state where the external artificial force just balances with the internal Coulomb repulsion. If the cooling power is not enough to overcome the stopband, it will no longer be possible to further improve the beam quality.

In order to form a crystalline structure, the beam temperature must be reduced typically to a milli-Kelvin (mK) range. The laser cooling method is the only mean to realize such an ultra-cold state [5]. In fact, the Doppler limit of the photon pressure is even below mK. At this temperature, if it can be reached, the betatron motion is completely frozen out, which means that the tune is depressed to zero; the beam has to pass through all the stopbands lying below the bare tune in order to reach the Doppler limit. It is thus practically important to know how severely the beam stability is affected through resonance crossing. A recent result of cooling experiments has actually demonstrated that the attainable beam temperature may be limited by some resonance effect [6].

2 COHERENT RESONANCES

2.1 Stopbands

Since a beam consists of a large number of interacting particles, its stability is definitely determined by the collective nature of the whole beam. The single-particle picture often gives us a good insight into underlying physics, but it is not accurate. The beam may be regarded as a sort of nonneutral plasma that possesses many collective oscillation modes. When the frequency of one mode, generally dependent on the beam density, fulfills a certain relation with the frequency of a particular harmonic in the external periodic driving force, the mode will be resonantly excited and then becomes unstable.

Provided that the momentum spread of the beam is negligible, the Hamiltonian of the betatron motion is given by [7]

$$H = \frac{p_x^2 + p_y^2}{2} + \frac{1}{2}(K_x x^2 + K_y y^2) + \frac{q}{p_0 \beta_0 c \gamma_0^2}\phi(x,y;s), \quad (1)$$

where K_x and K_y are the well-known transverse focusing functions that are periodic with respect to the path length s, q is the charge state of particles, β_0 and γ_0 are the Lorentz factors of the reference particle, and c is the speed of light. The scalar potential ϕ satisfies the Poisson equation

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\phi = -\frac{q}{\varepsilon_0}\iint f(x,y,p_x,p_y;s)\,dp_x dp_y, \quad (2)$$

where the distribution function f is a solution to the Vlasov equation

$$\frac{\partial f}{\partial s} + p_x \frac{\partial f}{\partial x} + p_y \frac{\partial f}{\partial y} - \frac{\partial H}{\partial x}\frac{\partial f}{\partial p_x} - \frac{\partial H}{\partial y}\frac{\partial f}{\partial p_y} = 0. \quad (3)$$

By solving these equations self-consistently, we can get all information about the collective behavior of the beam.

Following Hofmann et al. [8], we linearize the Vlasov-Poisson equations to perturbatively analyze the stability of the Kapchinsky-Vladimirsky (KV) distribution in a storage ring. Among a wide range of choices, we here adopt the lattice parameters of TARN II [9]. The TARN II ring, whose circumference is 77.7 m, has a six-fold symmetry. Each lattice period contains four bending magnets with an orbit curvature of 4.01 m. The growth rates of second- and fourth-order resonance instabilities in TARN II are plotted in Fig. 1 where a 1 MeV beam of $^{24}Mg^{+}$ ions has been assumed. The horizontal tune ν_x and vertical tune ν_y have been adjusted to be the same in this example; i.e. $\nu_x = \nu_y (\equiv \nu_0) = 1.8$. The abscissa represents the tune depression η defined as the ratio of the space-charge-depressed tune to the design tune ν_0. Third-order stopbands have not been indicated because the effects of odd-order resonances should be weak due to the even symmetry of the driving forces.

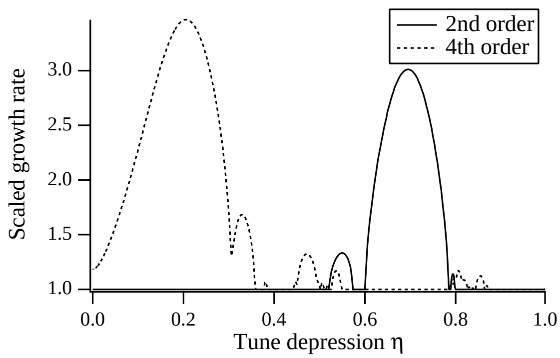

Figure 1: Resonance stopbands calculated from the 2D Vlasov-Poisson equations. The solid line expresses the growth rate of instability due to second-order resonances while the broken line that due to fourth-order resonances.

2.2 The Maintenance Condition

Sacherer proved that, if the charge density of a 2D beam maintains an elliptical symmetry, the root-mean-squared (rms) beam size satisfies the KV-type envelope equation regardless of the form of the distribution function. According to his theory [10], the horizontal and vertical rms beam sizes, i.e. a and b, obey the coupled differential equations

$$\begin{cases} \dfrac{d^2a}{ds^2} + K_x a - \dfrac{\varepsilon_x^2}{a^3} - \dfrac{K_{sc}}{2(a+b)} = 0, \\ \dfrac{d^2b}{ds^2} + K_y b - \dfrac{\varepsilon_y^2}{b^3} - \dfrac{K_{sc}}{2(a+b)} = 0, \end{cases} \tag{4}$$

where ε_x and ε_y are the horizontal and vertical rms emittances respectively, and K_{sc} is the beam perveance. The independence of Eq. (4) on the particle distribution strongly suggests that the second-order coherent resonances can be precisely analyzed with the rms envelope equation instead of with the complex Vlasov-Poisson equation system. We can actually demonstrate that the second-order stopbands derived from Eq. (4) are in perfect agreement with those predicted by the Vlasov theory.

The envelope equation allows us to give approximate formulae of the second-order coherent tunes. After straightforward algebra, we obtain

$$\nu_B = 2\nu_0 \sqrt{1 - \frac{\kappa}{\kappa + \sqrt{\kappa^2+1}}} \quad \text{(for breathing mode)},$$

$$\nu_Q = 2\nu_0 \sqrt{1 - \frac{3\kappa}{2(\kappa + \sqrt{\kappa^2+1})}} \quad \text{(for quadrupole mode)},$$

where we have put $\nu_x = \nu_y (\equiv \nu_0)$ and $\varepsilon_x \approx \varepsilon_y (\equiv \varepsilon)$ for the sake of simplicity, and $\kappa = RK_{sc}/8\nu_0\varepsilon$ with R being the average radius of the ring.

In an earlier simulation study with a MD code [1], it was pointed out that the following condition must be satisfied to realize beam crystallization:

$$\nu_0 < \frac{N_{sp}}{2\sqrt{2}}, \tag{5}$$

where N_{sp} is the superperiodicity of the ring. Shortly after this condition was proposed, people recognized that it is on the same basis of the second-order coherent resonance condition. In fact, at the space-charge limit where $\kappa \to \infty$, ν_B approaches $\sqrt{2}\nu_0$ while ν_Q gets equal to ν_0. For a storage-ring lattice with N_{sp}-fold symmetry, systematic linear stopbands appear when the envelope tune is around $nN_{sp}/2$ ($n = 1, 2, 3, \ldots$). Therefore, to avoid the possibility of crossing strong second-order resonances at very low temperature, the bare tune ν_0 must be in the range $\sqrt{2}\nu_0 < N_{sp}/2$ that leads to the condition (5). The inequality (5) has been often referred to as the *maintenance condition of a crystalline beam* as its importance was first emphasized in the stability study of ground-state structures.

3 SIMULATION RESULTS

Note that no particular information on the property of a crystalline beam is required to derive the maintenance condition. Considering that κ is initially close to zero rather than infinity, we suspect that the inequality (5) may be practically insufficient. Since both envelope tunes become equal to $2\nu_0$ when $\kappa = 0$, we expect that the condition above should be replaced by

$$\nu_0 < \frac{N_{sp}}{4}. \tag{6}$$

In order to verify this expectation, we performed systematic multi-particle simulations with a particle-in-cell (PIC) code. A transverse cooling effect was incorporated into the code through the simple transformation $p_{out} - p_{in} = -\zeta_\perp p_{in}$ where p_{in} and p_{out} are the transverse momenta before and after the cooling section, and $\zeta_\perp$ is the strength of the transverse cooling force. We consider again 1 MeV $^{24}Mg^+$ ions stored in TARN II with equal horizontal and vertical tunes. The initial normalized rms emittance is always set at 5.77×10^{-9} m· rad in both transverse directions. Since we also fix the line density of the beam at 1.5×10^6 m^{-1}, the initial tune depression is about 0.996 in all simulations. Figure 2 displays a typical time evolution of the normalized beam emittance when the cooling force is turned on. The result is based on the TARN II lattice with $\nu_0 = 1.8$. The initial particle distribution is the KV type, and the friction parameter has been chosen to be $\zeta_\perp = 0.0015$. If there is no obstacle to the cooling process, the emittance must be reduced until the tune depression η becomes nearly zero. However, Fig. 2 indicates that the reduction of η suddenly stopped at the value slightly above 0.8 after 3000 turns. The emittance behaviors similar to Fig. 2 were observed for non-KV initial beams as well. A possible explanation to this phenomenon is the tune locking caused by the interaction with a resonance stopband. The ASTRID group of Aarhus University has observed a similar effect in laser cooling experiments [6].

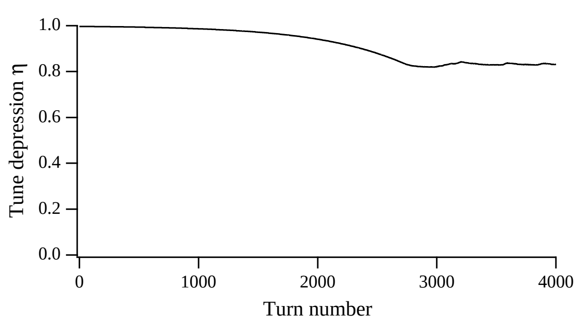

Figure 2: Time evolution of tune depression during a cooling process.

Figure 3 shows the lowest tune depressions achieved with various cooling rates $\zeta_\perp$. Three different types of initial phase-space distributions (KV, Gaussian, waterbag) have been taken into account. For reference, we have shaded the regions of the second- and fourth-order stopbands theoretically predicted by the Vlasov analysis. Figure 3(a), where the TARN II lattice with $\nu_0 = 1.8$ has been assumed, suggests that the cooling process was blocked by either a second- or a fourth-order stopband. At low cooling efficiency, the minimum tune depression appears to be limited by a fourth-order stopband. As we increase the cooling power, the value of η at which the tune locking occurs has been gradually lowered. However, even at $\zeta_\perp = 0.2$, we were not able to overcome a second-order stopband. Compared to this case, the result in Fig. 3(b) is essentially different. Specifically, the minimum tune depression achieved is much smaller than that in Fig. 3(a). The effective tune has been almost fully depressed unless the cooling rate is too low. As far as the fourth-order stopbands lying around $\eta \approx 0.7$ and $\eta \approx 0.4$ are concerned, we had no trouble going beyond them even with very weak cooling. Only the final stopband just above the density limit $\eta = 0$ produced some trouble as is evident from the picture. The trouble is, however, not serious since we can overcome it with a modest cooling force. Although the minimum η is not exactly zero in the region $\zeta_\perp > 0.02$, it is probably due to the limitation of the PIC algorithm. These results naturally lead us to the conclusion that a second-order stopband imposes severe restrictions on the minimum beam temperature reachable with laser cooling. Therefore, in order to form a large crystalline structure, the lattice of the storage ring should satisfy the condition (6) rather than the condition (5).

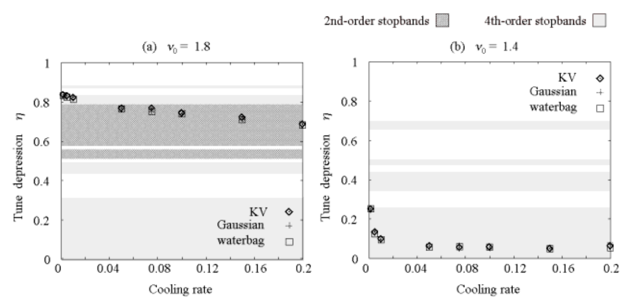

Figure 3: The lowest tune depression vs. cooling rate $\zeta_\perp$. The ordinate indicates the minimum tune depression at which the tune locking occurred. We have considered a 1 MeV $^{24}Mg^+$ beam cooled in the TARN II lattice.

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