ANALYTICAL FORMALISM FOR THE LONGITUDINAL ACCELERATION INCLUDING PARTICLE VELOCITY CHANGING EFFECT

Marc Doleans, Sang-ho Kim and Eugène Tanke

Spallation Neutron Source (SNS) Project

ORNL, Oak Ridge 37830, USA

Abstract

To calculate the longitudinal linear beam dynamics, the Panofsky equation which introduces the concept of transit time factor and average phase is widely used. The transit time factor is generally calculated under the assumption of a constant beta through the element. In the case of large beta variations or long and complex accelerating element, this approach can lead to some inaccuracies. To address this problem an analytic method taking into account the variation of the beta within the accelerating element has been developed. This method is applicable to any element by using decomposition of the electrical field into Fourier components. The average phase concept is adapted to the new formulation and the passage from the physical entrance phase to the average phase is clearly stated. The accuracy of the method is also presented through comparison with a slow and precise numerical approach.

1 INTRODUCTION

The desire to build cost effective accelerating structures favorises the emergence of high accelerating gradient and multi-gap cavities. As an example, the Spallation Neutron Source (SNS) will accelerate H^- ions with only two different types of 6-cell elliptical superconducting radio frequency (SRF) cavities. For such structures, the longitudinal dynamics treatment must offer flexibility to accommodate with the high accelerating gradient, with the large phase slips induced by the difference between the beta of the particles and the geometric beta of the structure, and with the field asymmetry present in the end-cells due to the large bore radius of the cavities. This need in flexibility is combined to the need in accuracy and fast computation. In the pursuit of these three prerequisites a new set of longitudinal dynamics equations have been developed. In Sec. 2, the usual set of equations for the longitudinal dynamics are applied to a case with large beta-changing and some losses in the accuracy for the energy gain and time of flight calculations are illustrated. In Sec. 3, a more general and precise method for the longitudinal dynamics based on consecutive analytical iterations is developed. In Sec. 4 this method is applied to the previous case of large beta-variation to show the gain in the accuracy.

2 APPROXIMATION OF BETA-CONSTANT

To treat the longitudinal dynamics of particles passing through an accelerating element, the beta of the particle is usually approximated constant. When the acceleration rate is small and the longitudinal field profile symmetric with respect to the geometric center, the element can efficiently be represented by drift spaces and thin gaps. For such a representation, a well-known set of equations has been derived [1] and constitutes a simple method to calculate the longitudinal dynamics. The basic thin gap transformation equations are

$$\begin{aligned} W_f &= W_i + qV_0T(k_i)\cos\phi_i \\ \phi_f &= \phi_i + \frac{qV_0}{2W_i}k_iT'(k_i)\sin\phi_i \end{aligned} \quad (1)$$

where W and ϕ designate the values of the kinetic energy and average phase of the particle, and the subscripts i and f refer to the initial and to the final values of these quantities, where q is the electrical charge of the particle, V_0 is the voltage across the gap, k is the wave vector, $T(k)$ is the transit time factor and $T'(k)$ its first derivative with respect to k.

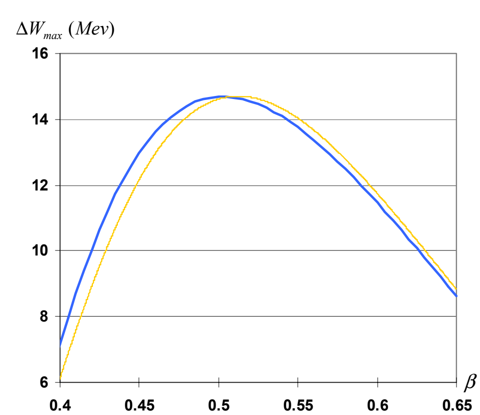

Figure 1: Maximum energy gain in function of the entrance beta. The dark curve is obtained with a step by step numerical method, the light curve is obtained using Eq. (1).

In the case of large variation of the beta and/or long and complex accelerating elements, the drift/thin-gap representation can lead to some loss of accuracy. As an example, a 5-cell cavity of geometric beta equal to 0.5, operating in pi-mode with an ideal sinusoidal longitudinal electric field profile is considered. Computation of the longitudinal dynamics is performed for H^- ions using the set of Eq. (1) and using a straightforward step-by-step numerical method for reference. From Eq. (1), the quantity $|qV_0T(k)|$ represents the maximum energy gain and this quantity is function of the entrance beta of the particle. In figure 1 the maximum energy gain profiles are plotted for both methods in the case of an extremely high accelerating gradient equal to 40 MV/m. For such a case, a discrepancy in the results of both methods appears due to the large variation of the particle's velocity within the element.

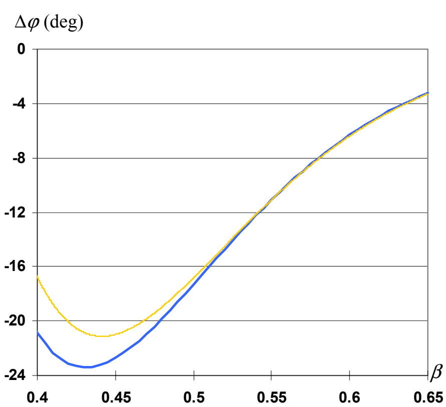

Figure 2: Difference in time of flight due to the variation of the beta. The dark curve is obtained from a step-by-step numerical method and the light curve from the set of Eq. (1)

As for the energy gain, the time of flight calculated with the set of Eq. (1) can lead to some inaccuracies. The variation in the time of flight, caused by the variations of the particle's velocity, are plotted for both methods in figure 2. As for the energy gain function, the loss in the accuracy is more pronounced when the variation of the beta inside the element is larger. To enhance the accuracy in the calculation of the longitudinal dynamics, other sets of equations have been derived from the basic treatment of Eq. (1). Some for example use numerical iterations [2] to find the value of the parameters at mid-gap. The benefit in the accuracy is nevertheless counter balanced by the complication of the method and the more expensive computation due to the introduction of numerical routines.

3 LONGITUDINAL DYNAMICS INCLUDING THE BETA-VARYING EFFECT

The energy gain and the phase for an on-axis particle passing from an initial location z_i to a location z in the accelerating element are defined by the integrals

$$\begin{aligned} \Delta W[z] &= q\int_{z_i}^{z} E[s]\cos(\varphi[s])\,ds \\ \varphi[z] &= \varphi[z_i] + \int_{z_i}^{z} k[s]\,ds \end{aligned} \tag{2}$$

where $E[s]$ is the longitudinal electric field at the location s. When the variation of the particle's beta is null, like in a drift space, the phase evolution written in Eq. (2) is linear, $\varphi_L[z] = \varphi[z_i] + k_{z_i}(z - z_i)$. A new phase variable can be defined as the difference of the total phase and its linear part, $\Delta\varphi[z] = \varphi[z] - \varphi_L[z]$. Since the variation of the beta within the accelerating element is usually a fraction of the entrance beta, it is of interest to develop the cosine term of Eq. (2) into a power series of the new phase variable $\Delta\varphi[z]$. Also, the variation of the wave vector k can be related to the energy gain $\Delta W[z]$ by expansion in power series. With these considerations a new set of equations can be developed for the longitudinal dynamics.

$$\begin{aligned} \Delta W[z] &= \sum_{m=0}^{\infty}\{\frac{q}{m!}\int_{z_i}^{z} E[s]\cos^{(m)}(\varphi_L[s])\,(\Delta\varphi[s])^m ds\} \\ \Delta\varphi[z] &= \sum_{n=1}^{\infty}\{\frac{1}{n!}k_{z=z_i}^{(n)}\int_{z_i}^{z}(\Delta W[s])^n ds\} \end{aligned} \tag{3}$$

Where $\cos^{(m)}(\varphi_L[z])$ refers the m^{th} derivative of the function $\cos(\varphi[z])$ with respect to $\varphi[z]$, evaluated for $\varphi[z] = \varphi_L[z]$, and where $k_{z_i}^{(n)}$ refers to the n^{th} derivative of k with respect to the particle energy, evaluated at the position z_i. The first term of the first series in Eq. (3) is equal to $\cos(\varphi_L[z])$ which corresponds to the case where the particle's beta does not vary. To solve the system of Eq. (3), a method based on consecutive analytical iterations is applied. To perform such analytical iterations, an analytic representation of the longitudinal electric field is needed and is done by developing the field in a series of cosine functions. For the first analytical iteration the case of zero energy variation is taken. Injecting this result into the second expression of Eq. (3) leads to a null phase variation. This result is reentered in the first expression of Eq. (3) to obtain a new expression for the energy gain function. The process is repeated as pictured in figure 3. The results of the consecutive iterations are subscripted with different indexes.

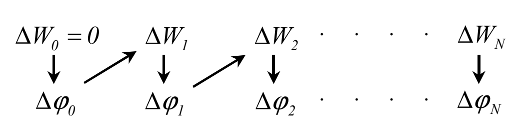

Figure 3: Principle of the analytical iteration process.

The initial parameters for the set of Eq. (3) are the amplitudes of the field components, the longitudinal boundaries z_i and z_f, the entrance energy of the particle W_i and the entrance phase of the particle $\varphi[z_i]$. Carrying the calculations it is possible to show that the expressions of the energy gain function and the total phase function can be expressed under the form

$$\begin{aligned} \Delta W[z] &= \sum_{n=0}^{N} \{ A_n cos(n\varphi[z_i]) + B_n sin(n\varphi[z_i]) \} \\ \Delta\varphi[z] &= \sum_{n=0}^{N} \{ C_n cos(n\varphi[z_i]) + D_n sin(n\varphi[z_i]) \} \end{aligned} \quad (4)$$

where the coefficients A_n, B_n, C_n, D_n depend on the field amplitudes, on z_i and z_f, and on W_i but not on the phase $\varphi[z_i]$. The set of Eq. (1) is an approximation of the more general set of Eq. (3) where the only term considered is the $n = 1$ term. The energy gain and the total phase are usually written as a function of an average phase Φ instead of the entrance phase $\varphi[z_i]$. The concept of average phase can be adapted to the formulation of Eq. (3). If the entrance phase corresponding to the maximum of the energy gain function is written $(\varphi[z_i])_{\Delta W_{MAX}}$, an average phase can be defined as $\Phi = \varphi[z_i] - (\varphi[z_i])_{\Delta W_{MAX}}$. With this definition, the energy gain function reaches its maximum when $\Phi = 0$. The set of Eq. (3) can be rewritten as

$$\begin{aligned} \Delta W[z] &= \sum_{n=0}^{N} \Delta W_n cos(n\Phi + \theta_n) \\ \Delta\varphi[z] &= \sum_{n=0}^{N} \Delta\varphi_n cos(n\Phi + \psi_n) \end{aligned} \quad (5)$$

with $\Delta W_n = (A_n^2 + B_n^2)^{1/2}$, $\Delta\varphi_n = (C_n^2 + D_n^2)^{1/2}$, $n\theta_n = arctan(B_n/A_n)$, and $n\psi_n = arctan(D_n/C_n)$.

4 ILLUSTRATION OF THE METHOD

In section 2 the example of a 5-cell cavity accelerating H^- with a gradient equal to 40 MV/m was chosen to show evidence of the beta-changing effect.

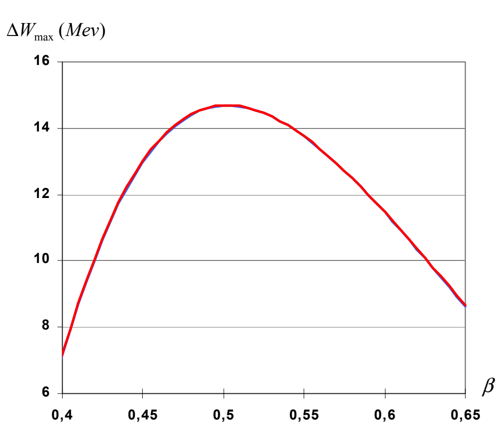

Figure 4: Maximum energy gain as a function of the entrance beta. The dark curve is obtained with a step by step numerical method, the light curve is obtained by using Eq. (3).

The same example is reused for the expressions obtained from the set of Eq. (3). The results presented are obtained after three analytical iterations and show a net gain in accuracy compared to the previous longitudinal dynamic treatment.

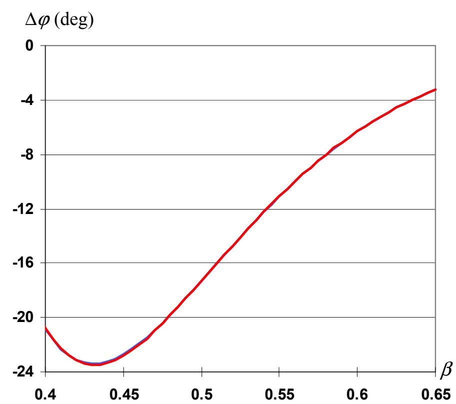

Figure 5: Difference in time of flight due to the variation of the beta. The dark curve is obtained from a step-by-step numerical method and the light curve from the set of Eq. (3)

5 CONCLUSION

The developed set of equations allows accurate calculations of the longitudinal dynamics even when the variation of the beta within the accelerating element is large. It does not require any symmetry of the electrical field and pre-computation of the coefficients depending on the accelerating element and on the entrance beta insure fast computation for the treatment of a bunch of particles. In the SNS medium beta superconducting cavities, the beta-varying effect is less severe but the longitudinal field profile is non-symmetric in the end-cells because of the large bore radius. The new method has been applied to this case and the accuracy is found to be excellent.

6 AKNOWLEDGEMENT

We are very thankful to all our collegues at SNS/ORNL for usefull comments and suggestions. This work is sponsored by the Division of Materials Science, U.S.Department of Energy, under contraction number DE-AC05-96OR22464 with UT-Battelle Corporation for Oak Ridge National Laboratory.

7 REFERENCES

[1] Thomas P. Wangler, “Principles of RF Linear Accelerators,” John Wiley & Sons, INC (1998)
[2] P. Lapostolle and M. Weiss, “Formulae and Procedures Useful for the Design of Linear Accelerators,” CERN-PS-2000-001 (DR) (2000)
[3] James H. Billen and Lloyd M. Young, “POISSON SUPERFISH,” LA-UR-96-1834, LANL (2000)