ORBIT HAMILTONIAN, SYNCHROTRON OSCILLATIONS AND SYNCHRO-BETATRON COUPLING

T. Suzuki, KEK, Tsukuba, Ibaraki 305-0801, Japan

Abstract

We develop a Hamiltonian formalism for synchro-betatron coupling using the orbit length as an independent variable. We start from a basic Hamiltonian in the curvilinear coordinate system and obtain symplectic equations of synchro-betatron coupled motin. This paper is an addition to a previous paper by the present author [1], where the part of synchrotron motion is stressed.

1 INTRODUCTION

Crowley-Milling and Rabinowitz [2] first discovered the synchro-betatron resonances driven by dispersion in rf cavities in NINA. They explained the mechanism of the resonances by considering the sudden change of energy by rf cavities that causes a sudden changes of the equilibrium orbit. Thus, betatron oscillations are excited. These are forced betatron oscillations driven by synchrotron oscillations. Piwinski and Wrulich [3] studied the counter effect of betatron oscillations on synchrotron oscillations due to the path lengthening by betatron oscillations. Then,the oscillations become coupled oscillations and the mechanism become symplectic. They described a complete theory of this effect.

Morton and Chao [4] and later Corsten and Hagedoorn [5] developed a Hamiltonian formalism for this this effect and derived a formula for the path lengthening by a simple canonical transformation. This term is also known as the CP (Central Position) phase in the theory of cyclotrons. (See the references in [5]) As an independent variable, Morton and Chao used the orbit length s and Corsten and Hagedoorn used the time t. The former authors treated a static case and the latter included acceleration.

The use of s (s-description) is more suitable than the use of t (t-description) in accelerator theories. All the devices are placed at fixed positions and observations are also done at fixed positions along the circumference of the accelerator. This s-description is necessary when we take into account the effects of localized objects. In the s-description, the effects can be described by a periodic δ-function. There is no easy way in the t-description. The localized nature of rf cavities is very important in synchro-betatron resonances. In this case, the resonances $\nu_x = n + m\nu_s$ are excited, where ν_x and ν_s are betatron and synchrotron tunes, and n,m are arbitrary integers. In a smooth, travelling wave approximation for rf, only the resonances $\nu_x = m\nu_s$ can be excited.

The t-description is usually used for synchrotron oscillations in an accelerated case [6]. This is certainly an approximatio. The concept of the rf bucket is also an approximation [7]. The present author [1] presented a symplectic theory of synchrotron oscillations and synchro-betatron coupling in the s-description. He sticked to a standing wave picture for rf and made a travelling wave approximation for analytical treatents only in the final stage. He followed a formalism by McMillan [8] and proved McMillan's equations of synchrotron motion. This s-description is important in the theory of synchro-betatron coupling because this description is usually used for betatron oscillations. In this paper, we add a Hamiltonian formalism to Ref. [1] because only equations of motion are used there though the author used only canonical variables.

2 ORBIT HAMILTONIAN

The orbit Hamiltonian in the s-description was given by Courant and Snyder. However, in their Hamiltonian, the components of the vector potential are not the ones in the direction of the coodinate axes. Some people call them as canonical components. Relations between the two components are given by Kolomensky and Lebedev [9]. This complication in the treatment by Courant and Snyder lies in the fact that they started from a Hamiltonian in the Cartesian coordinate system. If we introduce the Frenet-Serret coordinate system into the Lagrangian, derive a Hamiltonian in the t-description, and use the implicit function theorem as in Courant and Snyder, we obtain a Hamiltonian in the s-description that uses the components of the vector potential in the direction of the coodinate axes.

An alternative and more direct way is is to change the independent variable from t to s in the Lagrangian [10]. Then by the standard method, we obtain the orbit Hamiltonian. The same technique was used before by Teng [11] for betatron oscillations. When torsion is included, the Hamiltonian appears quite different from that given by Courant and Snyder. When the tortion is neglected, the Hamiltonian is

$$H_0 = -(1+x/\rho)\sqrt{(E/c)^2 - (mc)^2 - p_x^2} - e(1+x/\rho)A_s, \quad (1)$$

where we used conventional notations. This form is well-known. We put $A_x = A_y = 0$ and neglected the scalar potential. This is valid for the two-dimensional magnetic field studied in this paper. The canonical variables are (x, p_x) and $(t, -E)$.

Now, we study the vector potential of the two-dimensional magnetic field. By the Taylor expansion of $(1 + x/\rho)A_s$ and equating the coefficients by the relation $\nabla \times \mathbf{A} = \mathbf{B}$, we obtain an explicit and unique expression for A_s. The ambiguity due to gauge tansformation disappears because $\partial\chi/\partial s = 0$, where χ is a gauge function.

The explicit form is

$$(1 + x/\rho)A_s = A_0 - Bx - \frac{1}{2\rho}Bx^2 - \frac{1}{2}\frac{\partial B}{\partial x}x^2 \quad (2)$$

up to the second order in x. The vertical y motion is neglected. We use the (x, y, s) coordinate system here. The vetor potential for rf is

$$A_{rf} = -V\delta(s - s_0)\int^t \sin(\omega_{rf}t')dt' \quad (3)$$

The rf phase $\phi = \int^t \omega_{rf}dt$ is an expression for the travelling wave approximation. ϕ is $\omega_{rf}t$ in the standing wave picture. We assume that the rf cavities are placed in the straight section.

We put $t = t_0 + \tau$,where t_0 is the arrival time of the synchronous particle and τ is the time delay of an arbitrary particle. We further put $E = E_0 + \Delta E$, where E_0 is the energy of the synchronous particle and ΔE is the energy error. These are canonical transformations with generating functions $F = E(t_0 + \tau)$ and $F = -\tau(E_0 + \Delta E)$, respectively. $B(t)$ and $A_0(t)$ are expanded as $B(t_0) + \dot{B}(t_0)\tau$ and $A_0(t_0) + \dot{A}_0(t_0)\tau$, where the dot means $\partial/\partial t$.

We expand the Hamiltonian into a power series in ΔE and p_x and obtain up to the second order

$$\begin{aligned} H_1 \quad = \quad & \frac{\Delta E^2}{2c\beta_0^3\gamma_0^2E_0} - \frac{x}{\rho}\frac{\Delta E}{c\beta_0} + \frac{p_x^2}{2p_0} + \frac{1}{2}p_0Kx^2 + e\dot{B}\tau x \\ & + eV\delta_p(s - s_0)[\int^t \sin(\omega_{rf}t')dt' - \tau\sin\phi_0] \quad (4) \end{aligned}$$

Here, the relations

$$E_0' = -e\dot{A}_0 + eV\delta_p(s - s_0)\sin\phi_0 \quad (5)$$

and $p_0 = eB(t_0)\rho$ are used, where ϕ_0 is the synchronous phase. The prime denotes d/ds. The latter relation holds only at one point along the accelerator, but since the difference affects only the closed orbit, we neglect this differnce. Also, terms not containing canonical variables are neglected.

3 SYNCHRO-BETATRON COUPLED MOTION

The equations motion derived from the Hamiltonian in the previous section are

$$x' = p_x/p_0, \quad (6)$$

$$p_x' = \frac{p_0}{\rho}(\frac{\Delta E}{\beta_0^2E_0} - \frac{\dot{B}}{B}\tau) - \frac{eB}{\rho}x - e\frac{\partial B}{\partial x}x, \quad (7)$$

$$\tau' = \frac{1}{v_0}(\frac{x}{\rho} - \frac{\Delta E}{\beta_0^2\gamma_0^2E_0}), \quad (8)$$

$$\Delta E' = eV\delta_p(s - s_0)(\sin\phi - \sin\phi_0) + e\dot{B}x \quad (9)$$

These are the starting equations in Ref.[1]. Starting from these equations, the author explained the synchrotron oscillations and synchro-betatron coupled motion in the s-discription.

Since we mainly studied synchrotron part in the previous paper, we also study the betatron part. The equation of transverse motion is

$$x'' + \frac{p_0'}{p_0}x' + Kx = \frac{1}{\rho}(\frac{\Delta E}{\beta_0^2E_0} - \frac{\dot{B}\tau}{B_0}), \quad (10)$$

where

$$K = \frac{1}{\rho^2} + \frac{1}{B\rho}\frac{\partial B}{\partial x}$$

We can eliminate the inhomogenous term by introducing the dispersion function D as

$$x = x_\beta + D(\frac{\Delta E}{\beta_0^2E_0} - \frac{\dot{B}\tau}{B_0}). \quad (11)$$

Here, x_β is the displacement of betatron oscillations and is a canonical variable as we see later. In a static case without acceleration, we obtain

$$x_\beta'' + Kx_\beta = -D(\frac{\Delta E}{\beta_0^2E_0})'' - 2D'(\frac{\Delta E}{\beta_0^2E_0})'. \quad (12)$$

Synchrotron part is given in Ref.[1].

We now introduce a canonical transformation with a generating function that has old coordinates and new momenta.

$$F = F_1 + F_2 + F_3, \quad (13)$$

where

$$F_1 = \overline{p_\beta}\{x - D(\frac{\overline{\Delta E}}{\beta^2E_0} - \frac{\dot{B}\tau}{B_0})\},$$

$$F_2 = xD'p_0(\frac{\overline{\Delta E}}{\beta^2E_0} - \frac{\dot{B}\tau}{B_0}),$$

$$F_3 = -\overline{\Delta E}\tau - \frac{1}{2}DD'p_0(\frac{\overline{\Delta E}}{\beta^2E_0} - \frac{\dot{B}\tau}{B_0})^2.$$

Here, $\overline{p_\beta}$ is a canonical momentum conjugate to $\overline{x_\beta}$ and p_0 is the kinetic momentum. This generating function was first obtained by Morton and Chao with an approximation $D' = \dot{B} = 0$. Corsten and Hagedoor included the D'-term and the $\dot{B}$-term is now included.

The relations between the old and new variables are given as

$$x = \overline{x_\beta} + D\{\frac{\overline{\Delta E}}{\beta^2E_0} - \frac{\dot{B}}{B_0}(\overline{\tau} + \tau_\beta)\} \quad (14)$$

$$p_x = \overline{p_\beta} + D'p_0\{\frac{\overline{\Delta E}}{\beta^2E_0} - \frac{\dot{B}}{B_0}(\overline{\tau} + \tau_\beta)\} \quad (15)$$

$$\Delta E = \overline{\Delta E} + \frac{\dot{B}}{B_0}\tau_\beta\beta^2E_0 \quad (16)$$

$$\tau = \bar{\tau} + \tau_\beta \tag{17}$$

where

$$\tau_\beta = \frac{p_0 D' \bar{x}_\beta - \bar{p}_\beta D}{\beta^2 E_0} \tag{18}$$

The bars indicate the new canonical variables.

Though we can express a new Hamiltonian in terms of these new variables in a straight-forward way, the calculation is somewhat lengthy. We present hare a simpler static case without acceleration. This will be sufficient for practical calculations. The Hamiltonian is

$$\begin{aligned} H &= -\frac{\Delta\bar{E}^2}{2c\beta_0^3 E_0}(\frac{D}{\rho} - \frac{1}{\gamma_0^2}) + \frac{\bar{p_\beta}^2}{2p_0} + \frac{1}{2}p_0 K \bar{x_\beta}^2 \\ & -(eV/\omega_{rf})\delta_p(s-s_0)[\cos(\phi_0 + \omega_{rf}(\bar{\tau}+\tau_\beta)) \\ & +\omega_{rf}(\bar{\tau}+\tau_\beta)\sin\phi_0] \end{aligned} \tag{19}$$

Here, the equation $D'' + KD = 1/\rho$ is used. A similar expression is given by Corsten and Hagedoorn in the t-description.

We can further express the Hamiltonian by $\Delta\phi = \omega_{rf}\tau$ and $W = -\Delta E/\omega_{rf}$. Since ω_{rf} is constant in the static case, we can obtain the Hamiltonian and equations of motion by simply replacing the variables.

4 REFERENCES

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